Lecture 3: Pricing Methodology and Concept of Arbitrage

Revisiting the ingredients of a derivative contract

- underlying(s) (S), could be several
- expiration, or expiry, or maturity (T)
- strike (K)
- payoff function f(S) - how the final return depending on S
- main question: how to price the contract now?
- more subtle questions:
  - how does the derivative price depend on S? parallel move?
  - any other prominent factor affecting the price of derivative?
- Objective: relationship between the derivative price and S!

Starting with a call option example

- underlying S: stock price of certain company
- current S=$0.8
- expiration T=1 (year)
- strike K=$1
- call option payoff = max(S-K,0)
- scenarios at T=1:
  - if S(1) > 1, option payoff = S(1)-1
  - if S(1) <= 1, option expires worthless

Pricing approach 1: expectation

- assume zero interest rate
- suppose we have the future distribution of S(1)
  - S(1)=$3, with probability 0.3;
  - S(1)=$2, with probability 0.3;
  - S(1)=$1, with probability 0.2;
  - S(1)=$0.5, with probability 0.2
- shall the option price be the weighted average of the payoff (expectation given the distribution) = 0.3*2+0.3*1+0.4*0=0.9?
- flaw: the current option price is even higher than the current stock price!

Hedging strategy 1: minimize the potential loss

- sell the call
- buy the stock, so ready to deliver the stock if the call is exercised
- since I have the stock handy, no risk in the price escalating
- what would happen at T=1:
    - if S>1, deliver the stock for $1, no damage done, net gain = $0.1+$1=$1.1
    - if S<=1, let option expire, end up with $0.1 + whatever worth of the stock
- Either way, my net gain is between $0.1 and $1.1

Hedging strategy 2: stop-loss

- Buy the stock if the price crosses $1 upward;
- Sell the stock if the price crosses $1 downward;
- At expiration T=1
  - If the stock ends up above $1, we always have it ready;
  - If the stock ends up below $1, we come out clean
- Flaw: the above would have been a free hedge, except when S=1, you don't know if it is moving up (buy) or down (sell).
- Can we fix this? such as introducing a range? But high hedging cost!

What the models can help in hedging procedure?

- Cap the total amount that can be lost, damage control for the worst scenario
- Minimize the variance of the outcome
- Replicate the derivative: always precisely match the option payoff
- Avoid mispricing any risk
- **We are not expected to find a strategy to maximize the expected return!** It's what speculators do.

Pricing approach 2: delta hedging

- Focusing on price changes (stock and option)
- Notice that stock and call option always move in the same direction
- Suppose we know the ratio of price changes $\frac{\Delta V}{\Delta S}$, such as
    - S up \$0.1, V up \$0.04 so $\frac{\Delta V}{\Delta S} = 0.4$, or S down \$0.05, V down \$0.02,
- For each call option sold, we buy 2.5 shares of stock
- Putting together, the risk shall be eliminated, if price moves are small
- When changed to a new situation, reevaluate this ratio and update the positions
- Followed diligently, we should have a reasonably accurate cancellation of the risk!

How does this help us pricing options?

- Options can be replicated - their payoff almost exactly matched by the dynamically adjusted portfolio consisting of stocks (and maybe bank deposit)
- This match is realized in all possible scenarios
- Replicating portfolio and the option should have the same price!
- By looking at the composition of the portfolio (stock + bank deposits), we can price the replicating portfolio, therefore the option
- This is the idea behind Black-Scholes-Merton model
- Key to success: the ratio calculated, availability of the stock, etc.

Arbitrage

- The key statement: replicating (equivalent) portfolios should have the same price
- Why? because otherwise it will cause arbitrage
- What is arbitrage? getting positive return in excess to bank rate, without taking any risk
- No-arbitrage principle: under the assumptions of efficient market, there should be no arbitrage opportunities - “no free lunch”. If you want to generate high returns, you should be prepared to take some risk.
- Example: the previous stock option case if the price is $0.9.

What we plan to do in math finance

- List some assumptions, hope they are reasonable for the real world
- Suggest some underlying price dynamics model - must be stochastic
- Derive the stochastic model for the derivative price
- Combine the stock price and derivative price in the same setting, with proper weights - this is the price of the portfolio
- Choose the compositions (weights) so the portfolio does what you want - this is the main idea behind financial engineering
- No guarantee for the return, but the behavior of the portfolio will be curtailed

Assumptions

- No market reaction caused by your trading - quantities are reasonable, your transactions will not cause a stir in the market
- Liquidity: you can buy, or sell as many shares as possible, at the price quoted
- Shorting: you can sell securities you don’t own. The broker can borrow the security for you, sell on your behalf on the open market at the current price, but you need to buy back later to cover the borrow.
- Fractional quantities are allowed - quite reasonable
- No transaction costs

Using Arbitrage-free to price a forward contract

- A forward contract to exchange certain (K') yens for a dollar in 1 year
- Committed to exchange K' yens for a dollar, not an option
- In comparison with the ongoing exchange rate K
- The fair K' to put in the contract must be dependent on 1) the current exchange rate K, 2) the dollar interest rate r, and 3) the yen interest rate d
- Once signed, the contract remains effective till one year later, it may have positive, or negative values to the parties.
- This fair K', called the forward price, is an indicator of the market, therefore it also fluctuates in time.

Pricing the forward contract

- Suppose the current exchange rate is K, dollar interest rate r, and yen interest rate d
- Need to pay $1 in one year to exchange for receiving K' yens
- We can follow the steps in the strategy:
    - sell one-year riskless bond in dollar today, receiving $\frac{1}{1+r} < 1, \ (r > 0)$
    - exchange this amount to buy one-year riskless bond in yen: $\frac{K}{1+r}$
    - after one year, we will pay $1, receive $K\frac{1+d}{1+r}$ yens
- This implies that the arbitrage-free price is $K' = K\frac{1+d}{1+r}$

Dividends

- For certain stocks, payments to the holders are made on a regular basis
- The amount for each share may of may not be known beforehand
- The holder has choices of taking the payments or reinvesting
- The stock price will usually take a reduction
- For modeling convenience, dividend rate (d) are often used:
  - over the time period $(t, t + \Delta t)$
  - the dividend payment is $Sd\Delta t$
  - can be used to purchase $d\Delta t$ shares at t
  - accumulated over (0,t), the holder ends up a total of e^{dt} shares for each share

Continuously compounding effects

- The previous assumes an annual compounding
- If compounded n-times a year, the discount factor is $\left(1+\frac{r}{n}\right)^{-nt}$ where t is the number of years
- Continuous compounding ($n \to \infty$), the factor converges to e^{-rt}
- Using zero-coupon bond price: multiply the cashflow X at T by P(T) to arrive at the PV of X
- Theorem 2.1:
  - for a forward contract on S that pays a dividend d , with interest rate r
  - to buy the asset for K at expiration T
  - contract is worth $e^{-rT}\left(e^{(r-d)T}S_0 - K\right)$
  - in particular the contract will have zero value if and only if $K = e^{(r-d)T}S_0$
  - This is the forward price at time 0

Proof

- Arbitrage argument: the contract has zero value if $K = e^{(r-d)T} S_0$
  - Set up a portfolio: borrow $e^{-dT} S_0$ to buy e^{-dT} shares of the stock, total cost **zero**
  - At T, the loan becomes $e^{(r-d)T} S_0$, the number of stock shares is one
  - Net value at T is $S_T - e^{(r-d)T} S_0 = S_T - K$, the same as forward payoff, a replication
  - Since the price of setting this up is zero, the forward contract is worth zero.
- If the contract has another strike K', we write the payoff as a sum

$$S_T - K' = S_T - K + (K - K')$$

- The first part corresponds to a forward contract worth zero, and the second part is a fixed cashflow at T, with PV (present value)

$$e^{-rT}(K - K') = e^{-rT}\left(e^{(r-d)T} S_0 - K'\right)$$

Rigorous definition of arbitrage and a direct consequence

- Definition 2.1:

  A portfolio with value P(t) is said to be an *arbitrage portfolio*, if P(0) <= 0, but P(t) < 0 with zero probability, and P(t) > 0 with positive probability.

- In words, zero cost to set up, but impossible to lose (zero probability), possible to make money (positive probability). It's a situation of money making without taking any risk.

- No-arbitrage principle: arbitrage opportunities are not possible in an efficient market. An arbitrage opportunity would disappear as soon as it emerges, as other people will quickly learn about it, and the market force will drive up the price, so there will be no advantage left to set up such a portfolio.

- Theorem 2.2 (monotonicity) : If $P_A(T) \geq P_B(T)$ in **every** possible state, then for any $t < T$, $P_A(t) \geq P_B(t)$. If in addition the inequality holds at T in **some** state, then the inequality holds for *any* t<T.

Other Consequences

- Option prices are always strictly positive before expiration
- All **riskless** zero-coupon bonds with the same maturity should have the same price
- If two portfolios are of equal value today, and if at some future time, one is worth more than the other in **some** states, then the other must be worth more in **some other** states.
- Everything else being the same, the American option is worth as much as, or more than the European option. Adding extra rights to an option can only increase its value.
- American call on non-dividend-paying stock: always wait until T to exercise, therefore no extra value. Early exercise when S>K before T? Consider
  - payoff if exercised early: S(t)-K at t<T; or
  - keep the option, short the stock (receiving S(t) at t), then at T your call will allow you to buy the stock at S(T) if S(T)<K (letting the option expire), or K if S(T) >= K by exercising the option
  - comparison: receiving S(t), paying K at t, vs. receiving S(t) at t, paying min(K,S(T)) at T
  - second approach is to be preferred - the holder should not exercise early

Put-Call Parity

- Put-call parity: for the same strike, same maturity, the call and the put prices satisfy

$$C_t - P_t = S_t - e^{-r(T-t)}K = F_t$$

- If F(t)=0, or the stock price is close to the strike (assuming low interest rate)
  - Option in-the-money: positive payoff if exercised today
  - Option out-of-the-money: zero payoff if exercised today
  - Option at-the-money: cross-over point in between the above two, usually most popular with high volume. Used in most models to estimate volatility
- Using $C_t < S_t$, we can show $P_t \leq Ke^{-r(T-t)}$
- Call + K zero-coupon bonds with payoff $\max(S_T, K) \geq S_T$

$$C_t + Ke^{-r(T-t)} > S_t \qquad C_t > S_t - K$$

- Call > immediate exercise, can be used to prove that American call on non-div-pay stock should not be exercised prior to expiration

Rational Bounds

- Focus on the payoff functions
- Rational - no assumption on the dynamics of the underlying
- Theorem 2.7 (bounds on call options):

$$S_0 > C_0 > S_0 - e^{-rT}K = F_0$$

  - for the upper bound, use the stock to hedge
  - for the lower bound, compare the payoff of call with that of the forward

More results

- Time value of the options: two calls, same parameters except $T_1 < T_2$, we must have $C_1 \leq C_2$.
- Using the idea of American option to prove
- Two calls, same parameters except $K_1 < K_2$, we have

$$C_1 - C_2 \leq (K_2 - K_1)e^{-rT}$$

- C(K) is convex for t<=T:

$$\theta C(K_1) + (1-\theta)C(K_2) \geq C\left(\theta K_1 + (1-\theta)K_2\right)$$

Properties of call price

- Theorem 2.10 for call price C(K,T) on S
  - decreasing function of K
  - Lipshitz-continuous function of K, with Lipshitz constant Z(t,T)
  - convex function of K
  - for non-dividend-paying stock, an increasing function of T