Homework Problem Notes: Chapters 3-5

3.1 We should introduce $Y(t) = -X(t)$ and show that those requirements stated in the definition of Brownian motion are all met.

3.3 Here we consider the approximation model where for $t = n\Delta$

$$X(t) = X(0) + \sigma\sqrt{\Delta}\,(X_1 + \cdots + X_n)$$

So $X(1)$ is the sum of 10 i.i.d. variables. For part (c), we need the probability of $X_1 + X_2 + X_3 + X_4 + X_5 > 0$, and this corresponds to the event where more heads are turned up than tails (5H0T, 4H1T, or 3H2T). This probability is

$$p^5 + \binom{5}{4}p^4(1-p) + \binom{5}{3}p^3(1-p)^2.$$

3.4 As $S(t) = se^{X(t)}$, we have

(a)

$$\begin{aligned} P(S(1) > S(0)) &= P(X(1) - X(0) > 0) \\ &= P\left(\frac{X(1) - X(0) - \mu}{\sigma} > \frac{-\mu}{\sigma}\right) \\ &= 1 - \Phi\left(-\frac{\mu}{\sigma}\right) \\ &= 0.6915 \end{aligned}$$

(b)

$$\begin{aligned} P(S(2) > S(1) > S(0)) &= P(X(2) > X(1) > X(0)) \\ &= P(X(2) - X(1) > 0, X(1) - X(0) > 0) \\ &= P(X(2) - X(1) > 0) \cdot P(X(1) - X(0) > 0) \\ &= 0.6915^2 \\ &= 0.4781 \end{aligned}$$

(c)

$$\begin{aligned} P(S(3) < S(1) > S(0)) &= P(X(3) < X(1) > X(0)) \\ &= P(X(3) - X(1) < 0, X(1) - X(0) > 0) \\ &= P(X(3) - X(1) < 0) \cdot P(X(1) - X(0) > 0) \\ &= \Phi\left(-\frac{2\mu}{\sqrt{2}\sigma}\right) \cdot \left(1 - \Phi\left(-\frac{\mu}{\sigma}\right)\right) \end{aligned}$$

Here we used the Brownian motion property that increments over non-overlapping time intervals are independent.

3.7 Just use the formula for $P(T_y \leq t)$ on page 44 and pass the limit as $t \to \infty$. Make sure that you do this for $\mu \geq 0$ and $\mu < 0$ separately.

4.1 You need to find the approximate n such that

$$(1+r)^n = 3$$

The text suggests to use the approximation provided that r is small and n is not too small. It is actually more convenient just to take the log for both sides:

$$n = \frac{\log 3}{\log(1+r)}$$

which does not require the approximation conditions.

4.28 The rate of return r is defined as the solution to the equation

$$\frac{X_1}{1+r} + \frac{X_2}{(1+r)^2} = 100$$

We can solve the quadratic equation in r and obtain

$$r^* = \frac{X_1 + \sqrt{X_1^2 + 400X_2}}{200} - 1$$

Note that the other root is ignored since it is negative. Now we need to find $P(r^* > 0.1)$, that is the probability of

$$X_1 + \sqrt{X_1^2 + 400X_2} > 220$$

or

$$1.1X_1 + X_2 > 121$$

We note that $Y = 1.1X_1 + X_2$ is a normal random variable with mean 126 and variance $1.21 \times 25 + 25 = 2.21 \times 25$, so

$$P(Y > 121) = 1 - \Phi\left(\frac{121 - 126}{5\sqrt{2.21}}\right) = 0.7494$$

5.4 The payoff from the call is $\max(S(t) - K, 0)$ and the payoff from the underlying is $S(t)$. By comparing the payoff

$$\max(S(t) - K, 0) \leq S(t)$$

for any positive $S(t), K$, so the cost of owning one call must be less than the cost of owning one share of the underlying stock.

5.5 We can start from the put-call parity and notice that $P > 0$ to arrive at

$$S - C < Ke^{-rt}$$

5.7 As the payoff of a put is $\max(K - S(t), 0)$, the most it can be is K when $S(t)$ approaches zero. Since there is no reason for the strike to be set below the current stock price, it is possible that the payoff from the put can be higher than the current stock price. So it is not necessarily true that $P \le S$. On the other hand, as we just see that the payoff from a put is always less than or equal to K, that means that it is always true that $P \le K$.

5.12 The put-call parity for the standard call and put is irrelevant here. We notice that the sum of payoffs from the digital call and the digital put with the same strike is just the constant one. Owning a digital call and a digital put with the same strike is equivalent to have \$1 paid at the expiration. So

$$C_1 + C_2 = e^{-rt}$$

is the put-call parity for digital options.

5.15 Suppose you bought the put with strike K_1 and sold the put with strike K_2, at the expiration the payoff from this combination is

$$F(S(t)) = \begin{cases} K_1 - K_2 & S(t) \le K_2 \\ K_1 - S(t) & K_2 < S(t) \le K_1 \\ 0 & K_1 < S(t) \end{cases}$$

It is noted that $F(S(t)) \le K_1 - K_2$ for any $S(t)$ so the price of this combination, which is $P_1 - P_2$, must be less than $K_1 - K_2$.

5.17 (a) Always true. With longer expiration time, there is more room for the stock to vary and the call only benefits from the up moves so more fluctuation helps the value.

(b) Not always true. It depends on whether the US interest rate is higher or lower than the foreign currency interest rate.

(c) Always true. The reason is similar to part (a) as the put can benefit from more fluctuation.

5.24 The proof is similar to the case of call price in the text.

5.26 Assume $K_1 > e^{-r(t_2-t_1)}K_2$ and $S(t_1) > K_1$ so the question comes up if we should exercise now. Suppose we exercise the call with a payoff $S(t_1) - K_1$, which can be viewed as receiving $S(t_1)$ and paying K_1 at time t_1. Let us consider the following alternative: we short one share of the stock, receiving $S(t_1)$ at t_1, then by the time

t_2, we need to buy back one share of the stock to cover the short position. Since we own the call option at that time, we can buy back the stock at the price

$$\min(S(t_2), K_2) \leq K_2.$$

Comparing these two strategies, exercising at t_1 entails receiving $S(t_1)$ and paying K_1 both at t_1, while the other strategy allows you to receive $S(t_1)$ at t_1 and paying no more than K_2 at t_2, which is equivalent to $K_2 e^{-r(t_2-t_1)}$ paid at t_1. Then it is obvious that we would prefer the second strategy.

5.27 We can just compare these two payoff functions for $S(t)$ in three regions: (1) $S(t) < K$; (2) $K \leq S(t) < K + A$; and (3) $S(t) \geq K + A$.