# Area of Lattice Point Polygons

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The URL for these Beamer Slides: *"Pick's Theorem"*

`http://www.math.utah.edu/~treiberg/PickSlides.pdf`

- Ivan Niven and H. S. Zuckerman, Lattice Points and Polygonal Area, American Mathematical Monthly **7**4 (1967), 1195–1200; in Ann Stehny, Tilla Milnor, Joseph d'Atri and Thomas Banschoff, *Selected Papers on Geometry vol. 4*, Mathematical Association of America, 1979, pp. 149–153.
- Georg Pick, Geometrisches zur Zahlenlehre, Sitzungsberichte der deutschen naturalwissenshaftlich-medicinschen Verein für Böhma "Lotos" in Prag (Neue Folge) **19** (1899) 311–319.

# 4. Outline.

- Statement of Pick's Theorem
- Proof of Pick's Theorem
  - Translation, Rotation and Negation Invariace
  - Proof for Little Triangles
  - The Space of Lattice Polygons
  - Additivity of the Pick Function
  - Proof for Big Triangles
  - Proof for Convex Polygons
  - Proof for General Polygons
- Higher Dimensional Analogs

This gem of mathematics deserves to be known by all mathematics majors. It tells how to compute the area of a polygon just by counting lattice points that it contains. We follow Nivan and Zuckerman, but there are other slightly shorter proofs.

A simple polygon is one that is topologically equivalent to the circle. By lattice points in the plane we mean those points with integral coordinates.

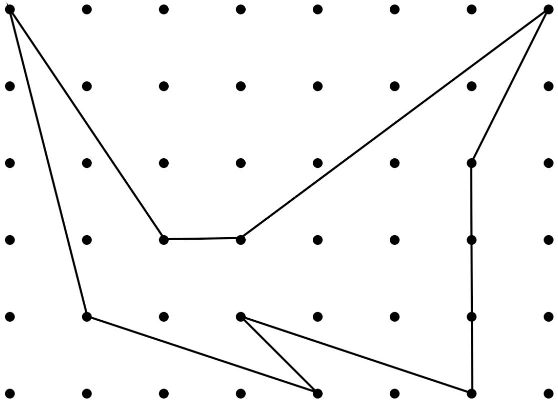
**Theorem (Pick 1899)**

*The area of a simple polygon $P$, all of whose points are lattice points is equal to the number of interior lattice points $I(P)$, plus half the number of lattice points on the boundary $\frac{1}{2}B(P)$, minus one*

$$A(P) = I(P) + \tfrac{1}{2}B(P) - 1.$$

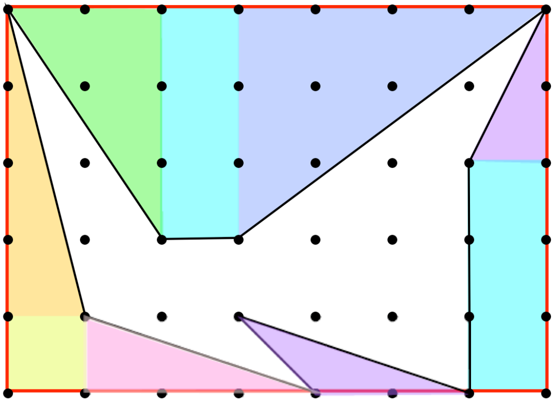
So for example, the area of a triangle that contains no lattice points other than its vertices is equal to $\frac{1}{2}$.

# 6. Example of Pick's Theorem



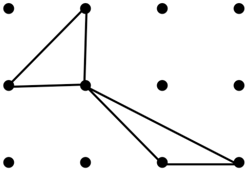
$$I = 9, \qquad B = 11,$$

$$A = I + \frac{1}{2}B - 1 = 9 + 5.5 - 1 = 13.5$$



$$
\begin{aligned}
A &= R - T - T - 2R - T - 2T - R - T \\
&= 35 - 2 - 3 - 6 - 6 - 2 - 1 - 1.5 \\
&= 13.5
\end{aligned}
$$

# 8. Pick's Theorem Fails if $P$ is Not Simple



$$I = 0, \quad B = 5, \quad A = 1$$

$$I + \tfrac{1}{2}B - 1 = 0 + 2.5 - 1 = 1.5$$

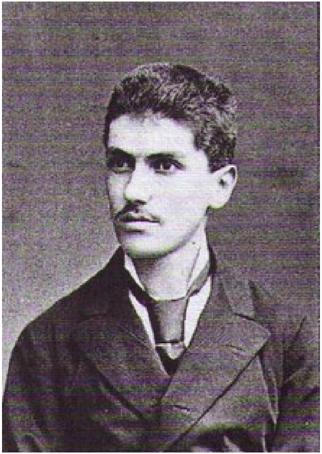


Figure: Georg Pick (1859–1942)

The Austrian Georg Pick completed his thesis at the University of Vienna under Königsberger and Weyr. Except for visiting Felix Klein in Leipzig in 1884, he worked his whole career at the Charles Ferdinand University in Prague. He returned to Vienna upon retirement in 1927. He died in the Theresienstadt Concentration Camp in 1942.

Pick wrote papers in differential geometry and complex analysis. He headed the committee to appoint Albert Einstein to the chair of mathematical physics in 1911. He introduced Einstein to the recent work by Ricci-Curbastro and Levi-Civita in curved manifolds, without which Einstein couldn't have formulated his theory of General Relativity of curved spacetimes.

# 10. Translation, Rotation and Negation Invariance

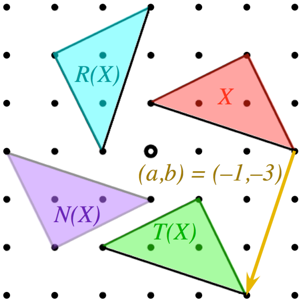



Figure: Translation $T$, Rotation $R$ and Negation $N$ of Polygon $X$

Applying the translation

$$T(x, y) = (x + a, y + b)$$

to each vertex $(x, y)$ of a polygon translates the polygon by vector $(a, b)$ and keeps its area the same. Similarly the transformations rotation and negation preserve the area

$$R(x, y) = (-y, x),$$
$$N(x, y) = (-x, -y).$$

A point of $(x, y)$ of the polygon $X$ is a lattice point if and only its transformation is a lattice point of the transformed polygon.

**Lemma**

*A lattice triangle $\Delta$ has area $\frac{1}{2}$ if and only if it contains no lattice points other than its three vertices.*

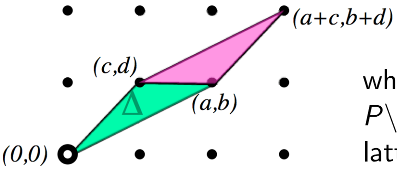



Figure: Triangle $\Delta$ and Parallelogram $P$

*Proof.* By translating a vertex to the origin, we may assume that the vertices of $\Delta$ are $(0,0)$, $(a,b)$ and $(c,d)$ such that the vectors $(a,b)$ and $(c,d)$ are not collinear. Let $P$ denote the parallelogram with vertices $(0,0)$, $(a,b)$, $(c,d)$ and $(a+c,b+d)$. Note that the triangle $P\backslash\Delta$ has vertices $(a+c,b+d)$, $(c,d)$ and $(a,b)$ which is the translate of the negative of $\Delta$

$$P\backslash\Delta = T(N(\Delta))$$

where $T(x,y) = (x+a+c, y+b+d)$. Thus $P\backslash\Delta$ has the same area as $\Delta$ and contains no lattice points other than its vertices.

It follows that $\Delta$ has no lattice points other than its three vertices if and only if $P$ has no lattice points other than its four vertices.

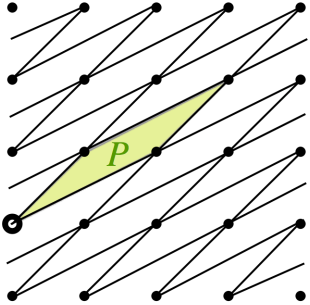



Figure: Plane tesselated by $P$

The entire plane can be covered by non-overlapping copies of $P$. Each copy is a translate by the vector

$$(ma + nc, mb + nd)$$

where $m, n$ are integers. The edges form a network $\mathcal{N}$ of lines, one set parallel to the $(0,0)$ $(a,b)$ edge of $P$ and the other parallel to the $(0,0)$ $(c,d)$ side. Call the set of points that are intersection points of these lines the vertices of $\mathcal{N}$. They are vertices of translates of $P$ so are lattice points. If some lattice point is not a vertex of $\mathcal{N}$, then it lies in one of the replicas of $P$ and is distinct from the vertices of the replica of $P$. Translating, we would then have a lattice point of $P$, not a vertex of $P$.

It follows that $\Delta$ has no lattice points other than its three vertices if and only if all lattice points are the vertices of $\mathcal{N}$.

# 13. Lattice Triangles Proof -

Since the vertices of $\mathcal{N}$ are the points $(ma + nc, mb + nd)$ where $m, n$ are integers, if follows that $\Delta$ has no lattice points other than its three vertices if and only if for every pair of integers $u, v$ there is a pair of integers $m, n$ such that $ma + nc = u$ and $mb + nd = v$.
Solving,

$$m = \frac{du - cv}{D}, \qquad n = \frac{av - bu}{D}, \qquad D = ad - bc.$$

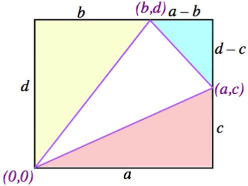
The fact that $(a, b)$ and $(c, d)$ are not collinear ensures $D \neq 0$. $D = \pm 1$ implies the existence of $m, n$. Suppose instead that $m, n$ exist for all $u, v$. taking first $u = 1$ and $v = 0$ and then $u = 0$ and $v = 1$ implies $D$ divides each of $a$, $b$, $c$ and $d$. Hence $D^2$ divides $ad - bc = D$. But this implies $D = \pm 1$.

It follows that $\Delta$ has no lattice points other than its three vertices if and only if $D = \pm 1$.

Area is given by the determinant, $A(\Delta) = \frac{1}{2}|D|$, so that $\Delta$ has no lattice points other than its three vertices if and only if the area of $\Delta$ is $\frac{1}{2}$. $\square$

**Theorem**

*Let $T$ be the triangle with vertices $(0,0)$, $(a,c)$ and $(b,d)$ then $A(T)=\frac{1}{2}|ad-bc|$.*




*Proof 1.* In case $0<b<c$ and $0<c<d$. Other cases similar.

$$A(T) = ad - \tfrac{1}{2}bd - \tfrac{1}{2}ac - \tfrac{1}{2}(a-b)(d-c)$$

$$= ad - \tfrac{1}{2}bd - \tfrac{1}{2}ac - \tfrac{1}{2}(ad-ac-bd+bc)$$

$$= \tfrac{1}{2}(ad-bc).$$

*Proof 2.* Using linear algebra, let $P$ be the parallelogram with vertices $(0,0)$, $(a,c)$, $(b,d)$ and $(a+b,c+d)$. Thus $P=L(S)$ where the transformation

$$L\binom{x}{y} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{x}{y}$$

and $S$ is the square with vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$. Thus

$$2A(T) = A(P) = \left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| A(S).$$

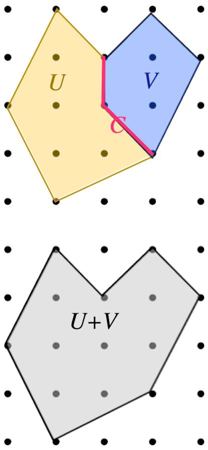



Figure: $U$, $V$ and $U+V$

Let $\mathcal{S}$ be the space of all simple polygons having all vertices as lattice points. If $U$ is in $\mathcal{S}$, then it is a polygon without holes and with a boundary that is a simple closed polygonal curve.

If $U, V \in \mathcal{S}$ are polygons such that the sets have in common a single connected part of the boundary of each, not just a single point, and no other points, then the union of the sets $U$ and $V$, minus the interior of their common boundary, forms a polygon $S$. Denote this polygon $U+V$. Note that $U+V$ is not defined for all pairs of $U$ and $V$ of $\mathcal{S}$, just for certain pairs.

Suppose that $f : \mathcal{S} \to \mathbf{R}$ is a function such that $f(U)+f(V)=f(U+V)$ for all $U, V \in \mathcal{S}$ for which $U+V$ is defined, then we say that $f(U)$ is an additive function.

**Theorem**

*Let $I(U)$ be the number of lattice points interior to $U$ not on the boundary and $B(U)$ be the number of lattice points on the boundary of $U$. Then the Pick Function*

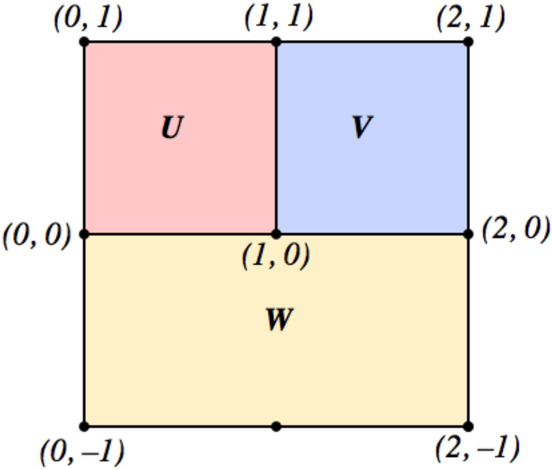
$$f(U) = \alpha I(U) + \beta B(U) + \gamma$$

*is additive if and only if $\beta = \alpha/2$ and $\gamma = -\alpha$.*

*Proof.* First suppose $f$ is additive. Let $U$ be the square $(0,0)\,(1,0)(1,1)\,(0,1)$, $V$ be the square $(1,0)\,(2,0)\,(2,1)\,(1,1)$ and $W$ be the rectangle $(0,-1)\,(2,-1)\,(2,0)\,(0,0)$. Then $U+V$ and $(U+V)+W$ are defined. Counting

$$\begin{aligned}
I(U) &= 0, & B(U) &= 4, & f(U) &= 4\beta+\gamma\\
I(V) &= 0, & B(V) &= 4, & f(V) &= 4\beta+\gamma\\
I(W) &= 0, & B(W) &= 6, & f(W) &= 6\beta+\gamma\\
I(U{+}V) &= 0, & B(U{+}V) &= 6, & f(U{+}V) &= 6\beta+\gamma\\
I((U{+}V){+}W) &= 1, & B((U{+}V){+}W) &= 8, & f((U{+}V){+}W) &= \alpha+8\beta+\gamma
\end{aligned}$$

# 17. Rectangles Used in the Proof.



Additivity implies

$$f(U) + f(V) = f(U + V)$$
$$f(U + V) + f(W) = f((U + V) + W)$$

Hence

$$\begin{cases} 8\beta + 2\gamma = 6\beta + \gamma \\ 12\beta + 2\gamma = \alpha + 8\beta + \gamma \end{cases} \implies \begin{cases} 2\beta + \gamma = 0 \\ -\alpha + 4\beta + \gamma = 0 \end{cases}$$

so $\beta = \alpha/2$ and $\gamma = -\alpha$.

Second, assume that $\beta = \alpha/2$ and $\gamma = -\alpha$ and $U, V \in \mathcal{S}$ such that $U + V$ is defined. Let $C$ be the common part of the boundary, and $k$ the number of lattice points in $C$. Since $C$ is more than a single point, it contains at least two lattice points, the ends of $C$. Any lattice point of $C$ other than an endpoint is an interior point of $U + V$. Hence

$$I(U + V) = I(U) + I(V) + k - 2.$$

Similarly, lattice points of $C$, except the endpoints are not boundary points of $U + V$ so

$$B(U+V) = B(U) - k + B(V) - k + 2 = B(U) + B(V) - 2k + 2.$$

Hence

$$\begin{aligned} f(U+V) &= \alpha(I(U) + I(V) + k - 2) + \beta(B(U) + B(V) - 2k + 2) + \gamma \\ &= (\alpha I(U) + \beta B(U) + \gamma) + (\alpha I(V) + \beta B(V) + \gamma) \\ &\quad + (k-1)(\alpha - 2\beta) - \alpha - \gamma \\ &= f(U) + f(V) \end{aligned}$$

since $\beta = \alpha/2$ and $\gamma = -\alpha$. $\square$

**Theorem**

*If $\Delta$ is any lattice triangle, then* $A(\Delta) =$

$$I(\Delta) + \frac{1}{2}B(\Delta) - 1.$$

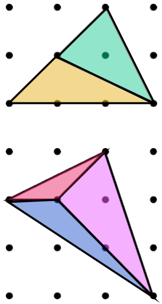


Figure: $\Delta_1 + \Delta_2$ and $\Delta_1 + \Delta_2 + \Delta_3$

*Proof.* In case $A(\Delta) = \frac{1}{2}$, we have seen $I(\Delta) = 0$ and $B(\Delta) = 3$ so the theorem holds in this case. The area of a general triangle is given by

$$A(\Delta) = \frac{1}{2}|(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)|$$

which is one half of an integer. We prove the theorem by induction on $A(\Delta)$ if we can show that $\Delta = \Delta_1 + \Delta_2$ or $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ where $\Delta_i$ are triangles of $\mathcal{S}$ having smaller area than $\Delta$.

If there is a lattice point other than a vertex on one of the sides of $\Delta$, then we break $\Delta$ into two triangles by means of the line from the point to the opposite vertex. If $\Delta$ has an interior lattice point, we connect the point to the three vertices and break $\Delta$ into three triangles. The additivity of the functions $A$ and $f$ completes the induction. $\square$

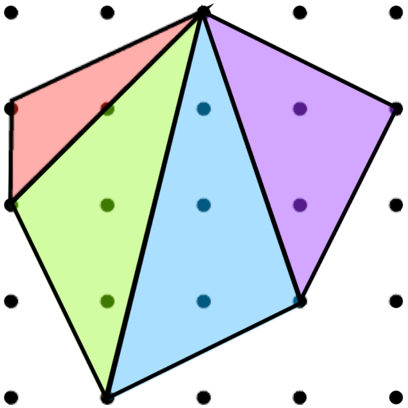


Figure: Convex Set is Sum of Triangles

**Theorem**

*If $P$ is any lattice convex polygon, then $A(P) = I(P) + \frac{1}{2}B(P) - 1$.*

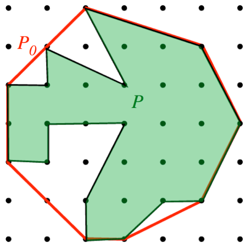
*Proof.* A convex polygon may be decomposed into triangles based at one of its vertices. Then use the additivity and area formula for triangles from the previous theorem. $\square$

In fact, a general simple lattice polygon $G$ can be decomposed into triangles. Thus the same argument shows the area formula for $G$. Instead, we give a simpler argument based on induction.

**Theorem**

*If $P$ is any simple lattice polygon, then $A(P) = I(P) + \frac{1}{2}B(P) - 1$.*

*Proof.* We argue by induction on the number of vertices $n$. If $n = 3$ then $P$ is a triangle then the formula holds for $P$. We may suppose $n > 3$ and that the theorem holds for all polygons with $n - 1$ or fewer vertices. Suppose that the vertices going around in order are $A_1, A_2, \ldots, A_n$ so that the sides are $A_1A_2, A_2A_3, \ldots, A_{n-1}A_n, A_nA_1$.




Let $P_0$ be the smallest convex set that contains $P$ (the convex hull of $P$). Then $P_0$ is a convex polygon in $\mathcal{S}$. The vertices of $P_0$ are a subset of $A_1, A_2, \ldots, A_n$. Some of the $A_i$ may be interior points of $P_0$ and some may be boundary points but not vertices of $P_0$. If all the vertices of $P$ are vertices of $P_0$ then $P = P_0$ so $P$ is convex and the result holds from the previous theorem.

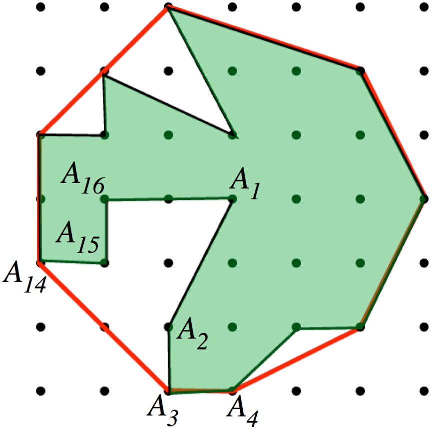



Figure: $n = 16$, $s = 3$, $r = 14$.

If $P$ is not convex, then some vertex, call it $A_1$ after a possible cyclic relabelling, is an interior point of $P_0$. Let $s \geq 2$ be the smallest subscript such that $A_s$ is on the boundary of $P_0$ and let $r \leq n$ be the greatest subscript such that $A_s$ is on the boundary of $P_0$.

Since $P_0$ has at least three vertices, at least three of the points $A_1, A_2, \ldots, A_n$ lie on the boundary of $P_0$, hence $r - s \geq 2$.
Thus, there is a vertex of $P$ strictly between $A_s$ and $A_r$.

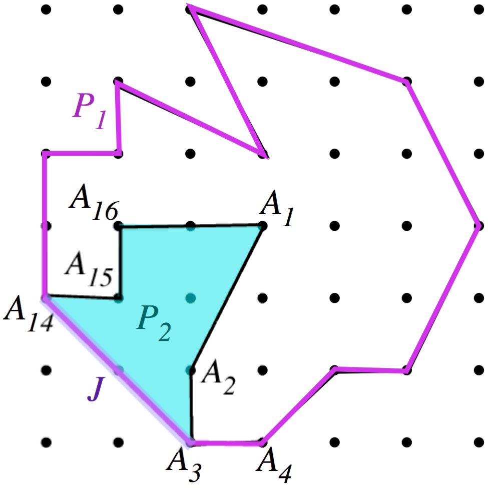



Figure: $P_1$ and $P_2$, $j = 3$.

Let $P_1$ be the polygon connecting $A_s, A_{s+1}, \ldots, A_r, A_s$. $P_1$ has fewer vertices than $P$ since $A_1$ is not a vertex of $P_1$.

Let $P_2$ be the polygon connecting $A_r, A_{r+1}, \ldots, A_n, A_1, \ldots, A_s, A_r$. $P_2$ has fewer vertices than $P$ because there is a vertex of $P$ strictly between $A_s$ and $A_r$. Thus the induction hypothesis applies to both $P_1$ and $P_2$.

By area additivity, $A(P_1) = A(P) + A(P_2)$. By the induction hypothesis,

$$
\begin{aligned}
A(P) &= A(P_1) - A(P_2) \\
&= I(P_1) + \tfrac{1}{2}B(P_1) - 1 - I(P2) - \tfrac{1}{2}B(P_2) + 1 \qquad (1) \\
&= I(P_1) - I(P2) + \tfrac{1}{2}B(P_1) - \tfrac{1}{2}B(P_2)
\end{aligned}
$$

Let $j$ be the number of lattice points on the segment $J$ from $A_r$ to $A_s$ in the boundary of $P_2$. These points are on the boundary of both $P_1$ and $P_2$. The remaining $B(P_2) - j$ boundary lattice points of $P_2$ are interior points of $P_1$. All interior points of $P_2$ and $P$ are also interior points of $P_1$. Hence

$$I(P_1) = I(P) + I(P_2) + B(P_2) - j. \qquad (2)$$

Of the boundary points of $P_1$, all are boundary points of $P$ except for the $j-2$ points of the segment $A_rA_s$ different from the points $A_r$ and $A_s$. All the rest of the boundary points of $P$ are precisely those boundary points of $P_2$ that don't lie on the segment $A_rA_s$. These number $B(P_2)-j$. Hence

$$B(P) = B(P_1) - j + 2 + B(P_2) - j \tag{3}$$

Inserting (2) and (3) into (1) we find

$$\begin{aligned} A(P) &= I(P_1) - I(P_2) + \tfrac{1}{2}B(P_1) - \tfrac{1}{2}B(P_2) \\ &= \big(I(P) + B(P_2) - j\big) + \tfrac{1}{2}\big(B(P) - B(P_2) + 2j - 2\big) - \tfrac{1}{2}B(P_2) \\ &= I(P) + \tfrac{1}{2}B(P) - 1. \quad \square \end{aligned}$$

Reeve has given a formula for volume of three dimensional polyhedra involving also the Euler-Poincaré characteristic from algebraic topology.

However, no formula involving only counts of lattice points on faces of $P$ can exist. In fact, no such formula exists for lattice tetrahedra (the convex hull of four lattice pints in three dimensions).

**Theorem**

*For three dimensional lattice tetrahedra $P$, there is no volume formula for $P$ of the form*

$$\alpha I(P) + \beta F(P) + \gamma E(P) + \delta W(P) + \eta = V(P). \tag{4}$$

*where $I(P)$ is the number of interior lattice points, $F(P)$ is the number of lattice points on the interior of the faces, $E(P)$ the number of lattice points on the edges excluding the vertices and $W(P)$ the number of vertices.*

*Proof.* Consider the three tetrahedra with given vertices.

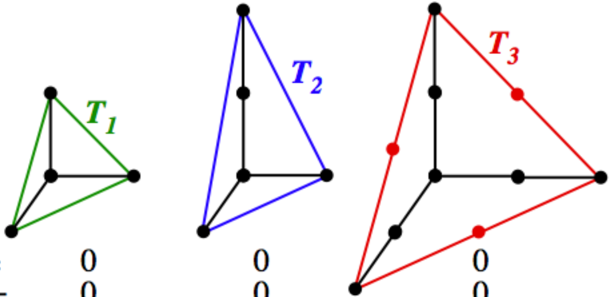
| | Vertices | $I$ | $F$ | $E$ | $W$ | $V$ |
|---|---|---|---|---|---|---|
| $T_1$ | (0,0,0), (1,0,0), (0,1,0), (0,0,1) | 0 | 0 | 0 | 4 | 1/6 |
| $T_2$ | (0,0,0), (1,0,0), (0,1,0), (0,0,2) | 0 | 0 | 1 | 4 | 1/3 |
| $T_3$ | (0,0,0), (2,0,0), (0,2,0), (0,0,2) | 0 | 0 | 6 | 4 | 4/3 |

Substituting into (4), results in the system of equations

$$\begin{aligned} 4\delta + \eta &= \tfrac{1}{6} \\ \gamma + 4\delta + \eta &= \tfrac{1}{3} \\ 6\gamma + 4\delta + \eta &= \tfrac{4}{3} \end{aligned}$$

which are inconsistent, so there is no solution for $\alpha, \beta, \gamma, \delta, \eta$. $\square$

# 29. Tetrahedra Used in the Proof.




| | $T_1$ | $T_2$ | $T_3$ |
|---|---|---|---|
| I = | 0 | 0 | 0 |
| F = | 0 | 0 | 0 |
| E = | 0 | 1 | 6 |
| W = | 4 | 4 | 4 |

Thanks!