

ON MINIMAL REPRESENTATIONS OF CHEVALLEY GROUPS OF TYPE D_n , E_n AND G_2

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ABSTRACT. Let G be a simply connected Chevalley group of type D_n , E_n or G_2 . In this paper, we show that the minimal representation of G is unique for types D_n and E_n and it does not exist for the type G_2 .

1. INTRODUCTION

Let F be a p -adic field with p odd. Let Φ be a simply laced root system (or of type G_2) and \mathfrak{g} the corresponding split semi-simple Lie algebra over the field F . Then there is a decomposition

$$\mathfrak{g} = (\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha) \oplus \mathfrak{t}$$

where \mathfrak{g}_α are one-dimensional root spaces and \mathfrak{t} a maximal split Cartan subalgebra. Let G be the corresponding simply connected Chevalley group. Let $B = TU$ be a Borel subgroup corresponding to a choice of positive roots Φ^+ . Here T is a maximal split torus which is described as follows. For every root α there is a homomorphism $\varphi_\alpha : \mathrm{SL}_2 \rightarrow G$ (the image will be denoted by $\mathrm{SL}_2(\alpha)$). Then T is generated by elements

$$\alpha^\vee(t) = \varphi_\alpha(\mathrm{diag}(t, t^{-1})),$$

for $t \in F^\times$. The map $\alpha^\vee : F^\times \rightarrow T$ is the co-root corresponding to α . Let Δ denote the set of simple roots. Recall that parabolic subgroups containing B are in one-to-one correspondence with subsets of Δ . For every subset $\Theta \subseteq \Delta$, there is a parabolic subgroup $P_\Theta = L_\Theta U_\Theta$ such that L_Θ is generated by T and $\mathrm{SL}_2(\alpha)$ for all α in Θ . In particular, $G = P_\Delta$ and $B = P_\emptyset$.

Any admissible representation V of G defines a character distribution χ in a neighborhood of 0 in \mathfrak{g} . Moreover, by a theorem of Howe and Harish-Chandra [HC], there exists a compact open subset Ω_V of 0 such that for every function f which is compactly supported in Ω_V ,

$$(1) \quad \chi(f) = \sum_{\mathcal{O} \in \mathcal{N}} c_{\mathcal{O}} \int \hat{f} \mu_{\mathcal{O}}.$$

Here \mathcal{N} is the set of nilpotent G -orbits in \mathfrak{g} , $\mu_{\mathcal{O}}$ is a suitably normalized Haar measure on \mathcal{O} , and \hat{f} is the Fourier transform of f with respect to the Killing form and a non-trivial

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character $\psi : F \rightarrow \mathbb{C}^\times$. Let

$$\mathcal{N}_V = \{\mathcal{O} \in \mathcal{N} \mid c_{\mathcal{O}} \neq 0\}.$$

The wavefront set $\text{WF}(V)$ of V is defined as the subset of \mathcal{N}_V consisting of all maximal elements in \mathcal{N}_V with respect to the partial order \leq defined in the following way:

$$\mathcal{O}_1 \leq \mathcal{O}_2$$

if and only if $\mathcal{O}_1 \subseteq \bar{\mathcal{O}}_2$ where $\bar{\mathcal{O}}$ denotes the topological closure of \mathcal{O} . The minimal orbit \mathcal{O}_{\min} is the smallest non-trivial nilpotent orbit in \mathfrak{g} . Its Bala-Carter [Ca] notation is A_1 . If α is a long root and X a non-zero element in \mathfrak{g}_α , then

$$\mathcal{O}_{\min} = \text{Ad}_G(X).$$

Definition. Suppose π is an irreducible admissible smooth representation of G such that the wavefront set of π is the minimal orbit, then we call π a *minimal* representation of G .

The main result of this paper is to determine minimal representations for groups of type D_n and E_n . See Theorem 1.1. In particular, we need to fix some notation for these two types roots systems. The set of simple roots is denoted by

$$\Delta = \{\beta_1, \beta_2, \dots, \beta_n\}.$$

We pick an indexing of simple roots so that $\beta_1, \beta_2, \beta_3$ and β_4 form the unique subdiagram of type D_4 , and

- The root β_2 corresponds to the branching point of the Dynkin diagram.
- The root β_1 is connected to β_2 only and to no other simple roots of G in the Dynkin diagram.

The last two simple roots β_3 and β_4 are picked, in no particular order, to complete the D_4 subdiagram. In terms of Bourbaki [Bo] notation, for type D_n groups, we have $\beta_1 = \alpha_n$ and $\beta_2 = \alpha_{n-2}$, and for type E_n groups, we have $\beta_1 = \alpha_2$ and $\beta_2 = \alpha_4$.

We define a character $\nu : F^\times \rightarrow \mathbb{C}^\times$ by $\nu(x) = |x|$. Given a character χ of T and a simple root β_i , we define a character $\chi_i : F^\times \rightarrow \mathbb{C}^\times$ by

$$\chi_i(t) = \chi(\beta_i^\vee(t)).$$

The main result of this paper is:

Theorem 1.1. *Let V be a minimal representation of G . Then V is the unique irreducible submodule of $\text{Ind}_B^G \chi$ (normalized induction) where χ is a character such that $\chi_i = \nu^{-1}$ for all $i \neq 2$ and χ_2 is the trivial character.*

Conversely, the unique irreducible submodule of $\text{Ind}_B^G(\chi)$ (where χ is as in Theorem 1.1) is a minimal representation with

$$c_{\mathcal{O}_{\min}} = 1.$$

This is Theorem 2.1 in [Sa]. Our next result deals with the exceptional group of type G_2 . In a sense this is the most interesting case. Indeed, a simple argument shows that a minimal representation of a split group of type D_n or E_n must be a representation of a linear group. If the type is B_3 , C_n or F_4 then a minimal representation must be a representation of a two-fold cover of a linear group (oscillator representation for C_n). A split group of type B_n for $n > 3$ has no minimal representation. However, if the type is G_2 then the situation is not so clear-cut. A minimal representation is either a representation of a linear group or a representation of a three-fold cover of the linear group. (See also a work of Torasso [To] for an explanation in terms of so-called admissible data.) Thus, for some time, it has remained somewhat a mystery whether a Chevalley group of type G_2 has a minimal representation. In [Ga] Gan showed that there is no minimal representation among spherical representations of G_2 . The following result now completely answers this question:

Theorem 1.2. *Let G be a Chevalley group of type G_2 . Then G has no minimal representation.*

As we have mentioned in the beginning of this introduction our results are subject to the condition $p \neq 2$. This restriction comes from the work of Mœglin and Waldspurger [MW]. Since [MW] makes use of the exponential map from \mathfrak{g} to G the restriction $p \neq 2$ appears to be unavoidable.

Methods of this paper are, of course, applicable to non-split groups. However, we have restricted ourselves to split groups for the following reasons. First, a classification of all non-split groups is quite complicated and, second, parameters of minimal representations may differ considerably from group to group (see [GS] for exceptional groups).

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2. PRINCIPAL SERIES REPRESENTATIONS

In this section we review some well known facts about principal series representations (see [Ro]) and prove that the induced representation $\text{Ind}_B^G(\chi)$ where χ is as in the statement of Theorem 1.1 has a unique irreducible submodule.

In this paper, Jacquet functors are normalized Jacquet functors as defined in Section 1.8(2)(b) in [BZ]. It is the left adjoint to the normalized induction functor.

Let E be an admissible representation of G . Now $J_U(E)$, the normalized module of U -coinvariants (Jacquet module) with respect to the maximal unipotent subgroup U , is finite dimensional. As a T -module, it can be decomposed as

$$J_U(E) = \bigoplus_{\chi} J_U(E)_{\chi}^{\infty}$$

where $J_U(E)_\chi^\infty$ consists of all v in $J_U(E)$ such that $(\pi(t) - \chi(t))^n v = 0$ for a sufficiently large n . The characters χ are called *exponents* of E . The Frobenius reciprocity implies that E is a submodule of an induced representation $\text{Ind}_B^G(\chi)$ if and only if χ is an exponent of E . Moreover, a character χ' is an exponent of $\text{Ind}_B^G(\chi)$ if and only if $\chi' = \chi^w$ for some w is in the Weyl group W of Φ . The multiplicity of an exponent χ is

$$\dim J_U(\text{Ind}_B^G(\chi))_\chi^\infty = |W_\chi|$$

where $W_\chi \subseteq W$ is the stabilizer of χ in the Weyl group W .

Proposition 2.1. *Let E be a submodule of $\text{Ind}_B^G(\chi)$ and β_i a simple root. Let s_i be the reflection defined by β_i . Recall that $\chi_i = \chi \circ \beta_i^\vee$.*

- (1) *If $\chi \neq \chi^{s_i}$ and $\chi_i \neq \nu^{\pm 1}$ then χ^{s_i} is also an exponent of E .*
- (2) *If $\chi_i = 1$ then $\dim J_U(E)_\chi^\infty \geq 2$.*

Proof. We shall prove both statements at once. The proof is a simple combination of representation theory for SL_2 and induction in stages. To that end, let $P_i = L_i U_i$ be the parabolic subgroup such that $[L_i, L_i] = \text{SL}_2(\beta_i)$. By representation theory of SL_2 , the conditions on χ in each of the two statements imply that $\text{Ind}_B^{P_i}(\chi)$ is irreducible. Since

$$\text{Ind}_B^G(\chi) = \text{Ind}_{P_i}^G(\text{Ind}_B^{P_i}(\chi)),$$

the Frobenius reciprocity implies that $\text{Ind}_B^{P_i}(\chi)$ is a quotient of $J_{U_i}(E)$. It follows that $J_U(\text{Ind}_B^{P_i}(\chi))$ is a quotient of $J_U(E)$. The proposition follows at once since the exponents of $\text{Ind}_B^{P_i}(\chi)$ are χ and χ^{s_i} if $\chi \neq \chi^{s_i}$ and χ with multiplicity 2 if $\chi = \chi^{s_i}$. \square

Corollary 2.2. *Let χ be a character of T such that $\chi_i = \nu^{-1}$ for all $i \neq 2$ and $\chi_2 = 1$. Then $\text{Ind}_B^G(\chi)$ has a unique irreducible submodule.*

Proof. Let $V' \oplus V''$ be a submodule of $\text{Ind}_B^G(\chi)$ such that $V' \neq 0$ and $V'' \neq 0$. Since $\chi_2 = 1$, the proposition implies that $\dim J_U(V')_\chi^\infty \geq 2$ and $\dim J_U(V'')_\chi^\infty \geq 2$. By exactness of the Jacquet functor, $\dim J_U(\text{Ind}_B^G(\chi))_\chi^\infty \geq 4$. On the other hand, it can be easily seen that W_χ , the stabilizer of χ in W , consist of only two elements: $W_\chi = \{1, s_2\}$. It follows that $\dim J_U(\text{Ind}_B^G(\chi))_\chi^\infty = 2$. This is a contradiction. \square

Our strategy of the proof of Theorem 1.1 is to show that any minimal representation has an exponent χ such that $\chi_i = \nu$ for all $i \neq 2$ and $\chi_2 = 1$.

3. WHITTAKER MODELS

We state a result of [MW] which relates wavefront sets and generalized Whittaker models of G . Let Y be an element in a nilpotent orbit \mathcal{O} of \mathfrak{g} . Let H be a semisimple element in \mathfrak{g} such that $[H, Y] = -2Y$ and all eigenvalues of H are integral. Existence of one such H is guaranteed by the Jacobson-Morozov theorem, but there are many

other choices, especially for Y in a small orbit. This observation is critical to us. Write $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ where \mathfrak{g}_i is the i -eigenspace of H . Let

$$\mathfrak{n}' = (Z_{\mathfrak{g}}(Y) \cap \mathfrak{g}_1) + \sum_{i \geq 2} \mathfrak{g}_i$$

and $N' = \exp(\mathfrak{n}')$. Let $\langle \cdot, \cdot \rangle$ denote the Killing form on \mathfrak{g} . The pair (Y, H) defines a character $\psi(Y, H)$ of N' by

$$\psi(Y, H)(\exp X) = \psi(\langle X, Y \rangle)$$

where X is in \mathfrak{n}' . If V is a representation of G , we set $J_{\psi(Y, H)}(V)$ to be the twisted Jacquet module with respect to the character $\psi(Y, H)$. Let

$$Wh_{\psi}(V) = \{\mathcal{O} \in \mathcal{N} \mid J_{\psi(Y, H)}(V) \neq 0 \text{ for some } H\}.$$

The following result is due to Moeglin and Waldspurger [MW].

Theorem 3.1. *Assume that $p \neq 2$. The wavefront set of V coincides with the set of maximal (with respect to the partial order \leq) nilpotent orbits in $Wh_{\psi}(V)$.*

Moeglin and Waldspurger also give a more precise description of $J_{\psi(Y, H)}(V)$ for Y in the wavefront set of V . There are two cases. The first case is when $\mathfrak{g}_1 = 0$. Then

$$(2) \quad \dim J_{\psi(Y, H)}(V) = c_{\mathcal{O}}$$

where $c_{\mathcal{O}}$ is given in (1). The second case, when $\mathfrak{g}_1 \neq 0$, is more complicated. Let $\mathfrak{n} = \bigoplus_{i > 0} \mathfrak{g}_i$ and $N = \exp(\mathfrak{n})$. Let \mathfrak{n}'' be the kernel of the functional $X \mapsto \langle X, Y \rangle$ where X is in \mathfrak{n}' . Let $N'' = \exp(\mathfrak{n}'')$. Then N/N'' is a Heisenberg group with the center N'/N'' . As such, it has a unique irreducible smooth representation W_Y with the central character $\psi(Y, H)$. Since N/N'' acts on $J_{\psi(Y, H)}(V)$ with the central character $\psi(Y, H)$, as an N/N'' -module, $J_{\psi(Y, H)}(V)$ is a multiple of W_Y and have

$$(3) \quad \dim \text{Hom}_N(W_Y, J_{\psi(Y, H)}(V)) = c_{\mathcal{O}}.$$

Finally we remark that for the given Y above, (2) or (3) will continue to hold for a different choice of H such that $[H, Y] = -2Y$.

We now describe some of our choices for H and Y . Let Y_i be a non-zero element of $\mathfrak{g}_{-\beta_i}$. Let H_{Δ} be in \mathfrak{t} such that $[H_{\Delta}, Y_i] = -2Y_i$ for all i . For any subset Θ of Δ , define

$$Y_{\Theta} = \sum_{\beta_i \in \Theta} Y_i.$$

According to the recipe given above the pair (H_{Δ}, Y_{Θ}) defines a character $\psi(Y_{\Theta}, H_{\Delta})$ of U . Moreover, let $P_{\Theta} = L_{\Theta}U_{\Theta}$ be the parabolic subgroup corresponding to Θ . We remind the reader that L_{Θ} is generated by T and $\text{SL}_2(\alpha)$ for all simple roots α in Θ . Note that

- $\psi(Y_{\Theta}, H_{\Delta})$ is trivial on U_{Θ} .
- $\psi(Y_{\Theta}, H_{\Delta})$ restricted to on $U \cap L_{\Theta}$ is a Whittaker functional for the group L_{Θ} .

For any representation V we have a natural isomorphism of vector spaces

$$(4) \quad J_{\psi(Y_{\Theta}, H_{\Delta})}(V) = J_{\psi(Y_{\Theta}, H_{\Delta})}(J_{U_{\Theta}}V)$$

where $J_{U_{\Theta}}$ is the space of U_{Θ} -coinvariants of V (Jacquet module). Thus, the above formula shows that if $J_{\psi(Y_{\Theta}, H_{\Delta})}(V) \neq 0$ then $J_{U_{\Theta}}(V) \neq 0$ and it is generic.

The rest of this section is devoted to a proof of Theorem 1.1. The proof consists of a series of lemmas. Let $P_1 = L_1U_1$ be the parabolic subgroup corresponding to β_1 . Let Y_1 be a non-zero element in $\mathfrak{g}_{-\beta_1}$. Then the pair (Y_1, H_{Δ}) defines the character $\psi(Y_1, H_{\Delta})$ of U which is trivial on U_1 . If V is a minimal representation then $J_{\psi(Y_1, H_{\Delta})}(V) \neq 0$ so the formula (4) shows that $J_{U_1}(V) \neq 0$ and it has generic (with respect to L_1) subquotients. The center of L_1 clearly contains elements $\alpha^{\vee}(t)$ for any root α perpendicular to β_1 . These include β_i for all $i \neq 1, 2$ and the root $\beta = \beta_1 + 2\beta_2 + \beta_3 + \beta_4$. By Schur's lemma elements of the center of L_1 have to act by a scalar on every irreducible subquotient of $J_{U_1}(V)$. The first result of this section is the following:

Lemma 3.2. *Let V be a minimal representation. Then $\beta_i^{\vee}(t)$ for $i \neq 1, 2$ acts trivially on any irreducible generic (with respect to L_1) subquotient of $J_{U_1}(V)$.*

Proof. The scalar by which $\beta_i^{\vee}(t)$ acts on an irreducible generic subquotient can be detected by a Whittaker functional (for L_1). Since every irreducible generic subquotient of $J_{U_1}(V)$ corresponds to a one-dimensional subquotient of $J_{\psi(Y_1, H_{\Delta})}$, via a Whittaker functional, it suffices to show that $\beta_i^{\vee}(t)$ acts trivially on $J_{\psi(Y_1, H_{\Delta})}(V)$. Let $P_{\Sigma} = L_{\Sigma}U_{\Sigma}$ be a parabolic subgroup corresponding to $\Sigma = \Delta \setminus \{\beta_1, \beta_2\}$. Let H be in \mathfrak{t} such that $[H, Y_i] = 0$ for all $i \neq 1, 2$ and $[H, Y_i] = -2Y_i$ for $i = 1, 2$. The pair (Y_1, H) defines a character $\psi(Y_1, H)$ of U_{Σ} . Since the restriction of $\psi(Y_1, H_{\Delta})$ to U_{Σ} is equal to $\psi(Y_1, H)$, we have a natural surjection

$$J_{\psi(Y_1, H)}(V) \twoheadrightarrow J_{\psi(Y_1, H_{\Delta})}(V).$$

If $i \neq 1, 2$ then $\langle \beta_i, \beta_1 \rangle = 0$ and the group $\mathrm{SL}_2(\beta_i)$ centralizes Y_1 and H . It follows that the action of G on V descends to an action of $\mathrm{SL}_2(\beta_i)$ on $J_{\psi(Y_1, H)}$. Since $J_{\psi(Y_1, H)}(V)$ is finite dimensional (by minimality of V) the action is trivial as $\mathrm{SL}_2(F)$ has no non-trivial finite dimensional representations. This proves the lemma. \square

Lemma 3.3. *Let V be a minimal representation. Then $\beta^{\vee}(t)$ acts by ν^2 on any irreducible generic subquotient of $J_{U_1}(V)$.*

We shall assume this lemma for a moment. Its proof is given towards the end of Section 4.

Lemma 3.4. *Let τ be an irreducible subquotient of $J_{U_1}(V)$. Then τ is not supercuspidal.*

Proof. Suppose τ is supercuspidal, in which case τ can be considered a quotient of $J_{U_1}(V)$. In particular,

$$V \subseteq \mathrm{Ind}_{P_1}^G(\tau).$$

Let $P_\Theta = L_\Theta U_\Theta$ be the parabolic subgroup corresponding to $\Theta = \{\beta_1, \beta_2\}$. Then we can write

$$\text{Ind}_{P_1}^G(\tau) = \text{Ind}_{P_\Theta}^G(\text{Ind}_{P_1}^{P_\Theta}(\tau)).$$

Next, by [BZ], $\text{Ind}_{P_1}^{P_\Theta}(\tau)$ is an irreducible generic representation of L_Θ , a reductive group of type A_2 . Recall that $Y_\Theta = Y_1 + Y_2$ where $Y_i \in \mathfrak{g}_{-\beta_i}$ and, by (4),

$$J_{\psi(Y_\Theta, H_\Delta)}(V) = J_{\psi(Y_\Theta, H_\Delta)}(J_{U_\Theta}(V)) \neq 0$$

since the generic L_Θ -module $\text{Ind}_{P_1}^{P_\Theta}(\tau)$ is a quotient of $J_{U_\Theta}(V)$. This is a contradiction since Y_Θ belongs to an orbit with Bala-Carter notation A_2 . The lemma follows. \square

In the following corollary we summarize what we have shown thus far:

Corollary 3.5. *Let V be a minimal representation. Then the character χ such that $\chi_i = \nu^{-1}$ for all $i \neq 1, 2$ and $\chi_1 \chi_2^2 = \nu^{-1}$ is an exponent of V .*

Proof. Indeed, we have shown that $J_U(V) \neq 0$ and there is a non-trivial T -invariant subquotient of $J_U(V)$ where $\beta_i^\vee(t)$ acts trivially for all $i \neq 1, 2$ and $\beta^\vee(t)$ acts as ν^2 . Since $\beta = \beta_1 + 2\beta_2 + \beta_3 + \beta_4$, and $\beta_3^\vee(t)$ and $\beta_4^\vee(t)$ act trivially, it follows that $\beta_1^\vee(t)\beta_2^\vee(t^2)$ acts as ν^2 as well. Since the modular function satisfies $\delta_B^{1/2}(\beta_i^\vee(t)) = \nu(t)$ it follows that χ is indeed an exponent of V . \square

As the corollary shows, we have reduced the problem of finding an exponent of the minimal representation V to figuring out what χ_2 is. The following lemma reduces to three possibilities.

Lemma 3.6. *Assume that V is a minimal representation and χ an exponent such that $\chi_i = \nu^{-1}$ for all $i \neq 1$ and 2. Then χ_2 is ν , 1 or ν^{-1} .*

Proof. Let $\Theta = \{\beta_3, \beta_4\}$ and $P_\Theta = L_\Theta U_\Theta$ the corresponding parabolic subgroup of G . Let $\chi' = \chi^{s_2}$. If $\chi_2 \neq \nu^{\pm 1}$ then χ' is also an exponent of V by Proposition 2.1. If we further assume that $\chi_2 \neq 1$ then

$$\chi'_3 = \chi_3/\chi_2 \neq \nu^{-1} \text{ and } \chi'_4 = \chi_4/\chi_2 \neq \nu^{-1}.$$

Using the induction in stages

$$\text{Ind}_B^G(\chi') = \text{Ind}_{P_\Theta}^G(\text{Ind}_B^{P_\Theta}(\chi'))$$

so $J_{U_\Theta}(V)$ maps to $\text{Ind}_B^{P_\Theta}(\chi')$ by Frobenius reciprocity. Since $\chi'_i \neq \nu^{-1}$ for $i = 3, 4$, any submodule V' of $\text{Ind}_B^{P_\Theta}(\chi')$ is L_Θ -generic. In particular, we can pick $Y_\Theta = Y_3 + Y_4$ with $Y_i \in \mathfrak{g}_{-\beta_i}$ for $i = 3, 4$ such that $J_{\psi(Y_\Theta, H_\Delta)}(V') \neq 0$. Using (4), it follows that

$$J_{\psi(Y_\Theta, H_\Delta)}(V) = J_{\psi(Y_\Theta, H_\Delta)}(J_{U_\Theta}(V)) \neq 0.$$

This contradicts the fact that V is minimal because Y belongs to an orbit with Bala-Carter notation $2A_1$. It follows that χ_2 must be ν^{-1} , 1 or ν . \square

The three cases $\chi_2 = 1, \nu$ and ν^{-1} will be referred to as cases a), b) and c) and need separate considerations.

Case a): $\chi_2 = 1$. Since $\chi_1\chi_2^2 = \nu^{-1}$, it follows that $\chi_1 = \nu^{-1}$ and χ is the exponent we have been looking for.

Case b): $\chi_2 = \nu$. Since $\chi_1\chi_2^2 = \nu^{-1}$, it follows that $\chi_1 = \nu^{-3}$. This exponent is eliminated by the following lemma.

Lemma 3.7. *A character χ of T such that $\chi_1 = \nu^{-3}$ and $\chi_2 = \nu$ cannot be an exponent of a minimal representation.*

Proof. Let $\Theta = \{\beta_1, \beta_2\}$ and $P_\Theta = L_\Theta U_\Theta$ be the corresponding parabolic subgroup of G . Now, if V is a submodule of $\text{Ind}_B^G(\chi)$ then, using the induction in stages and Frobenius reciprocity, there is a non-trivial map (of L_Θ -modules) from J_{U_Θ} to $\text{Ind}_B^{P_\Theta}(\chi)$. We need to understand this L_Θ -module. To this end, realize the root subsystem spanned by roots β_1 and β_2 in the space of triples (x, y, z) such that $x + y + z = 0$ and the simple roots are $\beta_1 = (1, -1, 0)$ and $\beta_2 = (0, 1, -1)$. Let $\text{SL}_3 = [L_\Theta, L_\Theta]$. Then the restriction of the unramified character χ of T to $T \cap \text{SL}_3$ can be identified with a triple $\chi = (x, y, z)$ so that

$$\chi(\beta_i(t)) = |t|^{\langle \chi, \beta_i \rangle}.$$

Under this identification the character χ such that $\chi_1 = \nu^{-3}$ and $\chi_2 = \nu$ is represented by $\chi = (-\frac{5}{3}, \frac{4}{3}, \frac{1}{3})$. Notice that this character is regular for the A_2 -root system. This implies that $\text{Ind}_B^{P_\Theta}(\chi)$ has a unique irreducible submodule. Furthermore, since $\chi_2 = \nu$, the induction in stages through the parabolic subgroup corresponding to β_2 implies that $\text{Ind}_B^{P_\Theta}(\chi)$ has a generic submodule V_g and a degenerate quotient V_d . Both are irreducible by Rodier [Ro]. One could also deduce this from Theorems 2.2 and 3.5 in [Ze]. It follows that the image of $J_{U_\Theta}(V)$ in $\text{Ind}_B^{P_\Theta}(\chi)$ must contain V_g . Using (4), it follows that

$$J_{\psi(Y_\Theta, H_\Delta)}(V) = J_{\psi(Y_\Theta, H_\Delta)}(J_{U_\Theta}(V)) \neq 0.$$

This contradicts the fact that V is minimal because Y belongs to an orbit with Bala-Carter notation A_2 . \square

Case c): $\chi_2 = \nu^{-1}$. Since $\chi_1\chi_2^2 = \nu^{-1}$, it follows that $\chi_1 = \nu$. Theorem 1.1 follows from the following lemma:

Lemma 3.8. *Let V be a minimal representation. Assume that V has an exponent χ' such that $\chi'_i = \nu^{-1}$ for all $i \neq 1$ and $\chi'_1 = \nu$. Then the character χ such that $\chi_i = \nu^{-1}$ for all $i \neq 2$ and $\chi_2 = 1$ is also an exponent of V .*

Proof. Notice that $\chi' = \chi^{s_1}$. By the assumption, V is a submodule of $\text{Ind}_B^G(\chi')$. Again, we use the parabolic $P_\Theta = L_\Theta U_\Theta$ such that $\Theta = \{\beta_1, \beta_2\}$. The induction in stages and the Frobenius reciprocity imply that $J_{U_\Theta}(V)$ has a non-trivial quotient contained in $\text{Ind}_B^{P_\Theta}(\chi')$. It remains to understand this L_Θ -module. The restriction of χ' to $T \cap \text{SL}_3$ corresponds to $\chi' = (\frac{1}{3}, -\frac{2}{3}, \frac{1}{3})$. Since $\chi'_2 = \nu^{-1}$ induction in stages through the parabolic

subgroup corresponding to β_2 implies that $\text{Ind}_B^{P_\Theta}(\chi)$ has a degenerate submodule V_d and a generic quotient V_g .

We claim that V_d and V_g are irreducible. To see this it suffices to show that they are irreducible as $\text{SL}_3 = [L_\Theta, L_\Theta]$ -modules. To that end the restriction of χ' and χ to $T \cap \text{SL}_3$ corresponds to $\chi' = (\frac{1}{3}, -\frac{2}{3}, \frac{1}{3})$ and $\chi = (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$ respectively. By Example 11.2 in [Ze], $\text{Ind}_B^{P_\Theta}(\chi')$ is a direct sum of two irreducible submodules. Hence V_d and V_g are irreducible and proves our claim.

We will give an alternative proof of the claim in which we will also compute the exponents. Since V_d is fully induced, it is a straightforward exercise to show that the exponents of V_d are χ' and $\chi = (\chi')^{s_1}$ twice. Further decomposing of V_d would imply that it contains, as a subquotient, a module with only one exponent. But there are only two such representations for SL_3 : Steinberg and the trivial representation. Their exponents are $(-1, 0, 1)$ and $(1, 0, -1)$, respectively. This is clearly a contradiction which shows that V_d is irreducible. This argument shows that V_g is also irreducible.

In view of minimality of V , the non-trivial quotient of $J_{U_\Theta}(V)$ in $\text{Ind}_B^{P_\Theta}(\chi')$ must be equal to V_d . This shows that χ is an exponent of V . The lemma and main theorem are proved at last. \square

4. HEISENBERG GROUPS

This section is devoted to the proof of Lemma 3.3. In words, we want to calculate the action of $\beta(t)$ on $J_{\psi(Y, H_\Delta)}(V)$ where Y is a non-zero element in \mathfrak{g}_{β_1} and $\beta = \beta_1 + 2\beta_2 + \beta_3 + \beta_4$. This will be accomplished by comparing $J_{\psi(Y, H_\Delta)}(V)$ with $J_{\psi(Y, H)}(V)$ where H belongs to a Jacobson-Morozov triple $\{X, H, Y\}$ generating $\mathfrak{sl}_2(\beta_1)$. Recall how $\psi(Y, H)$ is defined. First, the element H defines a gradation of $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$. In order to describe \mathfrak{g}_i , let

$$S_i = \{\alpha \in \Phi \mid \langle \alpha, \beta_1 \rangle = i\}.$$

Then, for every $i \neq 0$, the space \mathfrak{g}_i is a direct sum of \mathfrak{g}_α for all α in S_i . Since $\langle \alpha, \beta_1 \rangle \leq 2$ and is equal to 2 only if $\alpha = \beta_1$, it follows that \mathfrak{g}_2 is one dimensional, spanned by X , and $\psi(Y, H)$ is a character of the one-dimensional subgroup $N' = \exp(\mathfrak{g}_2)$. There is more to this story, however. Let $\mathfrak{n} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. This is a two step (Heisenberg) nilpotent algebra with the center \mathfrak{g}_2 . The normalizer of \mathfrak{n} in \mathfrak{g} is a parabolic subalgebra $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{n}$ where

$$\mathfrak{m} = \mathfrak{g}_0 = \mathfrak{t} \oplus (\bigoplus_{\alpha \in S_0} \mathfrak{g}_\alpha).$$

Let $Q = MN$ denote the parabolic subgroup in G with the Lie algebra \mathfrak{q} . Of course, N is a Heisenberg group with the center N' . As such, N has a unique irreducible representation (π_Y, W_Y) with central character $\psi(Y, H)$. Next, note that $J_{\psi(Y, H)}(V)$ is the maximal quotient of V such that N' acts via $\psi(Y, H)$ on it. In particular, as an N -module, $J_{\psi(Y, H)}(V)$ is a multiple of W_Y and we have an isomorphism of N -modules (see [We])

$$(5) \quad W_Y \otimes \text{Hom}_N(W_Y, J_{\psi(Y, H)}(V)) \cong J_{\psi(Y, H)}(V)$$

defined by $v \otimes A \mapsto A(v)$. In fact, this is also an isomorphism of $[M, M]$ -modules under the following actions. First of all, by the usual construction of the Weil representation, there is an action, also denoted by π_Y , of a double cover of $[M, M]$ on W_Y . In fact, as it was shown in [KS], this action descends down to $[M, M]$. Second, the action of $[M, M]$ on V descends down to $J_{\psi(Y,H)}(V)$ and is denoted by π . Putting things together, an element m in $[M, M]$ acts on an element A in $\text{Hom}_N(W_Y, J_{\psi(Y,H)}(V))$ by $\pi^{-1}(m) \circ A \circ \pi_Y(m)$.

If V is minimal then $\text{Hom}_N(W_Y, J_{\psi(Y,H)}(V))$ has finite dimension $c_{\mathcal{O}_{\min}}$ by (3) and the action of the perfect group $[M, M]$ must be trivial. It follows that the action of $[M, M]$ on $J_{\psi(Y,H)}(V)$ can be reconstructed from the action π_Y of $[M, M]$ on W_Y . In order to exploit this idea we need a *polarization* of N/N' to write down W_Y . Define

$$S_1^+ = \{\alpha \in \Phi^+ : \langle \alpha, \beta_1 \rangle = 1\} \text{ and } S_1^- = \{\alpha \in \Phi^- : \langle \alpha, \beta_1 \rangle = 1\}.$$

Let \mathfrak{n}^+ and \mathfrak{n}^- be the direct sums of \mathfrak{g}_α with α in S_1^+ and S_1^- , respectively. Then $\mathfrak{g}_1 = \mathfrak{n}^+ \oplus \mathfrak{n}^-$ such that $[\mathfrak{n}^+, \mathfrak{n}^+] = 0$ and $[\mathfrak{n}^-, \mathfrak{n}^-] = 0$. In particular, W_Y has a realization as $S(\mathfrak{n}^-)$, the space of locally constant and compactly supported functions on \mathfrak{n}^- . Explicit formulas for the action π_Y of $\text{SL}_2(\beta) \subset [M, M]$ on $S(\mathfrak{n}^-)$ were worked out in [KS]. Roughly speaking, as it is shown in the proof of Proposition 2 in [KS], there is a polarization \mathfrak{n}_β^- of \mathfrak{g}_1 which is $\text{SL}_2(\beta)$ -invariant. Then the action $\text{SL}_2(\beta)$ on $S(\mathfrak{n}_\beta^-)$ is by translations. This is correct, without any sign ambiguities, since $\text{SL}_2(\beta)$ is a perfect group by Proposition 1 in [KS]. The action of $\beta^\vee(t)$ on $S(\mathfrak{n}^-)$ is obtained from the action of $\beta^\vee(t)$ on $S(\mathfrak{n}_\beta^-)$ via a partial Fourier transform. We need the following very special case: For every f in $S(\mathfrak{n}^-)$

$$(\pi_Y(\beta^\vee(t))f)(0) = |t|^{-\frac{\langle \beta, \lambda \rangle}{2}} f(0)$$

where $\lambda = \sum_{\alpha \in S_1^-} \alpha$. This formula gives the action of $\beta^\vee(t)$ on the delta functional $\delta(f) = f(0)$. On a case by case basis one easily verifies that $\langle \beta, \lambda \rangle = -4$ in each case. It follows that the action of $\beta^\vee(t)$ on δ is given by

$$(6) \quad \delta(\pi_Y(\beta^\vee(t))f) = |t|^2 \delta(f).$$

Proof of Lemma 3.3. Recall that $\psi(Y, H_\Delta)$ is a character of U and its restriction to Z is equal to $\psi(Y, H)$. It follows that $J_{\psi(Y, H_\Delta)}(V)$ is a quotient of $J_{\psi(Y, H)}(V)$. Using the isomorphism (5) and $W_Y \cong S(\mathfrak{n}^-)$ we have a surjection

$$S(\mathfrak{n}^-) \otimes \text{Hom}_N(S(\mathfrak{n}^-), J_{\psi(Y, H)}(V)) \rightarrow J_{\psi(Y, H_\Delta)}(V).$$

Recall that $N^+ = \exp(\mathfrak{n}^+)$ is contained in U and note that the character $\psi(Y, H_\Delta)$ is trivial on N^+ . The maximal quotient of $S(\mathfrak{n}^-)$ such that N^+ acts trivially on it is one-dimensional and spanned by the delta function δ . Thus the above surjection descends to a surjection

$$\mathbb{C} \cdot \delta \otimes \text{Hom}_N(S(\mathfrak{n}^-), J_{\psi(Y, H)}(V)) \rightarrow J_{\psi(Y, H_\Delta)}(V).$$

(This map is in fact an isomorphism since $\text{Hom}_N(S(\mathfrak{n}^-), J_{\psi(Y, H)}(V))$ and $J_{\psi(Y, H_\Delta)}(V)$ have the same dimension, equal to the coefficient $c_{\mathcal{O}_{\min}}$ in the character expansion (1)

of V - we do not need this, however.) The action of $\beta^\vee(t)$ on δ is by $|t|^2$ by (6) and is trivial on $\text{Hom}_N(S(\mathfrak{n}^-), J_{\psi(Y,H)}(V))$. The proof of Lemma 3.3 is now complete.

5. G_2

Let G be a Chevalley group of type G_2 over the local field F . In this section we show that G has no minimal representation. The proof is similar in nature to the proof of uniqueness of the minimal representation for simply laced groups. We use a variety of degenerate Whittaker models to narrow down parameters of a possible minimal representation.

Let $\Delta = \{\beta_1, \beta_2\}$ be a set of simple roots for G_2 such that β_1 is long and β_2 is short. Let $P_1 = L_1U_1$ be the parabolic subgroup corresponding to β_1 . Let Y_1 be a non-zero element in $\mathfrak{g}_{-\beta_1}$. The minimal orbit is generated by Y_1 . The pair (Y_1, H_Δ) defines the character $\psi(Y_1, H_\Delta)$ of U which is trivial on U_1 . If V is a minimal representation then $J_{\psi(Y_1, H_\Delta)}(V) \neq 0$ so the formula (4) shows that $J_{U_1}(V) \neq 0$ and it has generic subquotients.

Let $\beta = \beta_1 + 2\beta_2$. Note that β is perpendicular to β_1 . It follows that $\beta^\vee(t)$ is in the center of L_1 . In fact, if we identify $L_1 \cong \text{GL}_2$ as in [Mu], then $\beta^\vee(t)$ is a scalar matrix:

$$\beta^\vee(t) = \text{diag}(t, t).$$

By Schur's lemma elements of the center of L_1 must act by a scalar on every irreducible subquotient of $J_{U_1}(V)$.

Lemma 5.1. *Let V be a minimal representation. Then, up to a complex number of norm one, $\beta^\vee(t)$ acts by ν^2 on any irreducible generic subquotient of $J_{U_1}(V)$.*

Proof. The proof of this is completely analogous to the proof of Lemma 3.3 and involves a comparison of $J_{\psi(Y_1, H_\Delta)}(V)$ and $J_{\psi(Y_1, H_1)}(V)$ where Y_1 and H_1 belong to an \mathfrak{sl}_2 -triple (X_1, H_1, Y_1) spanning $\mathfrak{sl}_2(\beta_1)$. In the simply laced case, however, working out the action of $\beta^\vee(t)$ on the Heisenberg representation W_Y is based on the fact that there is a polarization invariant for $\text{SL}_2(\beta)$. There is no such polarization here, so this is why we have a weaker result here. \square

Lemma 5.2. *Let τ be an irreducible subquotient of $J_{U_1}(V)$. Then τ is not supercuspidal.*

Proof. Assume that τ is supercuspidal. Then τ can be considered a quotient of $J_{U_1}(V)$. In particular,

$$V \subseteq \text{Ind}_{P_1}^G(\tau)$$

where induction is not normalized. Using the identification $L_1 \cong \text{GL}_2$, the previous Lemma implies that $\tau = \sigma \otimes |\det|$ where σ has a unitary central character. The square root of the modular character of L_1 acting on U_1 is

$$\delta_1^{1/2}(g) = |\det(g)|^{5/2}.$$

Let $I_1(s, \sigma)$ denote the (normalized) induced representation, where we induce the representation $\sigma \otimes |\det|^s$ on L_1 . We have

$$V \subseteq I_1(-3/2, \sigma).$$

Now, if V is minimal, then it is clearly a proper submodule of the induced principal series. Thus the principal series $I_1(s, \sigma)$ reduces for $s = -3/2$. On the other hand, Shahidi [Sh] has shown that if σ is a supercuspidal representation with unitary central character then $I_1(s, \sigma)$ could reduce only for a half integral point between -1 and 1 . Since $-3/2$ is outside this range, we have a contradiction. The lemma is proved. \square

The previous lemma shows that any minimal representation V is induced from a Borel subgroup. In particular, $J_{U_2}(V) \neq 0$, where $P_2 = L_2U_2$ is the parabolic subgroup corresponding to β_2 . We identify L_2 with GL_2 as in [Mu]. Let Y_2 be a non-zero element in $\mathfrak{g}_{-\beta_2}$. The minimal orbit does not contain Y_2 . The pair (Y_2, H_Δ) defines a character $\psi(Y_2, H_\Delta)$ of U which is trivial on U_2 . Note that the minimal orbit does not contain Y_2 . Thus, if V is a minimal representation, $J_{\psi(Y_2, H_\Delta)}(V) = 0$. The formula (4) shows that irreducible subquotients of $J_{U_2}(V)$ are one-dimensional characters of L_2 . It follows that

$$V \subseteq I_2(s, \chi \circ \det)$$

where I_2 is a degenerate (normalized) principal series (denoted by I_α in [Mu].) and χ a unitary character. The representation $I_2(s, \chi \circ \det)$ is irreducible unless

$$\chi = 1, s = \pm 3/2, \text{ or } \chi^2 = 1, s = \pm 1/2 \text{ or } \chi^3 = 1, s = \pm 1/2.$$

In order to describe irreducible subquotients of $I_2(s, \chi \circ \det)$ we need some notation. Let $\pi(\mu_1, \mu_2)$ be the tempered principal series representation of GL_2 where μ_1 and μ_2 are two unitary characters. Let $\delta(\chi)$ be the Steinberg representation of GL_2 twisted by the character $\chi \circ \det$.

The following description of non-trivial subquotients of $I_2(s, \chi \circ \det)$ is taken from Section 4 in [Mu]. (Note that $I_2(-s, \chi \circ \det)$ has the same irreducible subquotients as $I_2(s, \chi^{-1} \circ \det)$, so it suffices to consider s positive.)

Proposition 5.3. *In $R(G_2)$, the Grothendieck group of admissible representations of G_2 , we have:*

(1) *Let χ be of order 2. Then*

$$I_2(1/2, \chi \circ \det) = J_1(1, \pi(1, \chi)) + J_1(1/2, \delta(\chi)).$$

(2) *Let χ be of order 3. Then*

$$I_2(1/2, \chi \circ \det) = J_1(1, \pi(\chi, \chi^{-1})) + J_2(1/2, \delta(\chi^{-1})).$$

(3) *Let $\chi = 1$. Then*

$$I_2(1/2, 1_{\mathrm{GL}_2}) = \pi(1) + J_1(1, \pi(1, 1)) + J_1(1/2, \delta(1)).$$

(4) Let $\chi = 1$. Then

$$I_2(3/2, 1_{\mathrm{GL}_2}) = 1_{\mathrm{G}_2} + J_1(5/2, \delta(1)).$$

where $J_i(s, \sigma)$ is the unique (Langlands) quotient of the principal series representation $I_i(s, \sigma)$ and $\pi(1)$ a discrete series representation which is a submodule of $I_1(1/2, \delta(1))$.

Now it is easy to see that none of the subquotients described above is a minimal representation. As seen in the proof of Lemma 5.2 a minimal representation V can be a submodule of $I_1(s, \sigma)$ where σ has a unitary central character only if $s = -3/2$. Thus, if we look the case $\chi^3 = 1$, for example, then $J_1(1, \pi(\chi, \chi^{-1}))$ cannot be minimal since $J_1(1, \pi(\chi, \chi^{-1}))$ is a submodule of $I_1(-1, \pi(\chi^{-1}, \chi))$. The same argument applies to all Langlands quotients of type J_1 appearing in the above Proposition and to $\pi(1)$. Finally, $J_2(1/2, \delta(\chi^{-1}))$ cannot be minimal since it is a submodule of $I_2(-1/2, \delta(\chi))$ and, by Frobenius reciprocity, the space of U_2 -coinvariants of $J_2(1/2, \delta(\chi^{-1}))$ is L_2 -generic. This shows that G has no minimal representations.

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