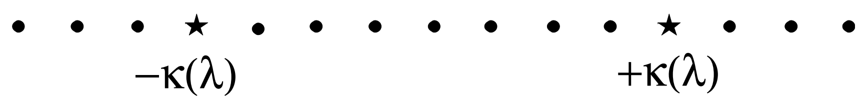
Some special discrete groups of linear transformations

W. Rossmann

Character Table

$SL(2,\mathbb{R})$	$SO(2)$
$FD(\lambda)$	$\sum_{-\kappa(\lambda)}^{+\kappa(\lambda)} e^{ikx} = -\frac{e^{i\lambda x} - e^{-i\lambda x}}{e^{ix/2} - e^{-ix/2}}$
$DS_{\pm}(\lambda)$	$\sum_{\kappa(\lambda)}^{\infty} e^{\pm ikx} = \frac{\pm e^{\pm \lambda x}}{e^{ix/2} - e^{-ix/2}}$
$PS(\lambda)$	$\sum_{-\infty}^{\infty} e^{ikx} = \delta(x)$



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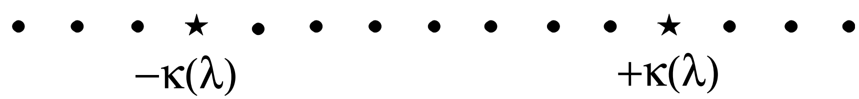
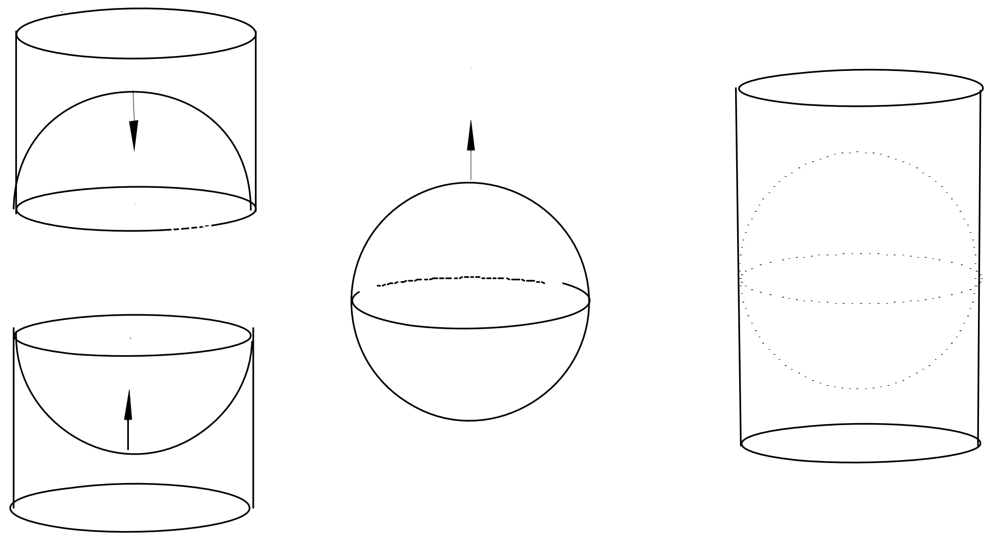


$$\mathsf{PS} = \mathsf{DS}_{-} + \mathsf{FD} + \mathsf{DS}_{+}$$

Orbit Contour Integrals



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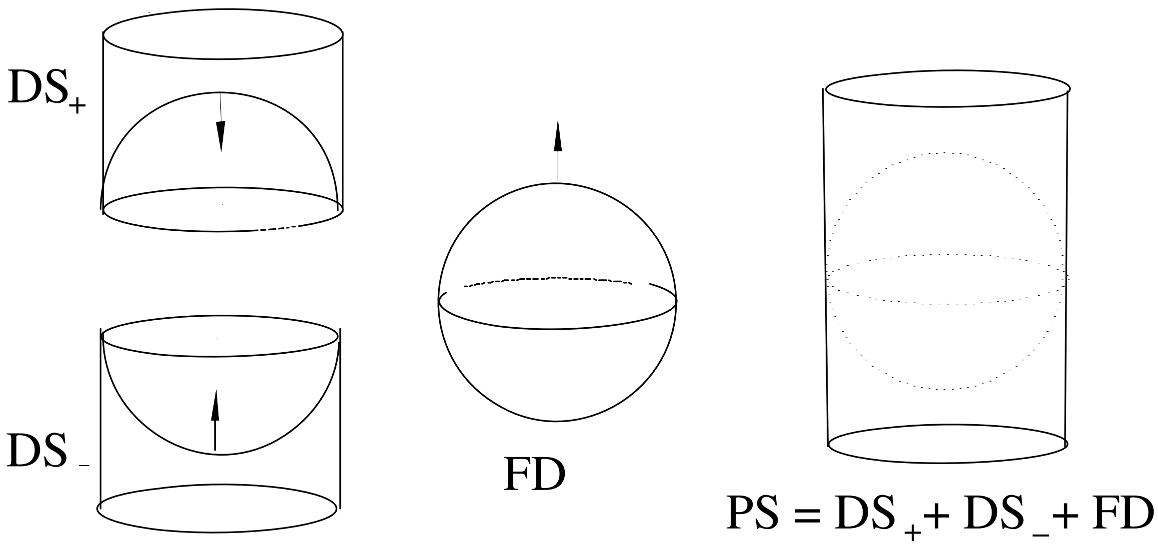
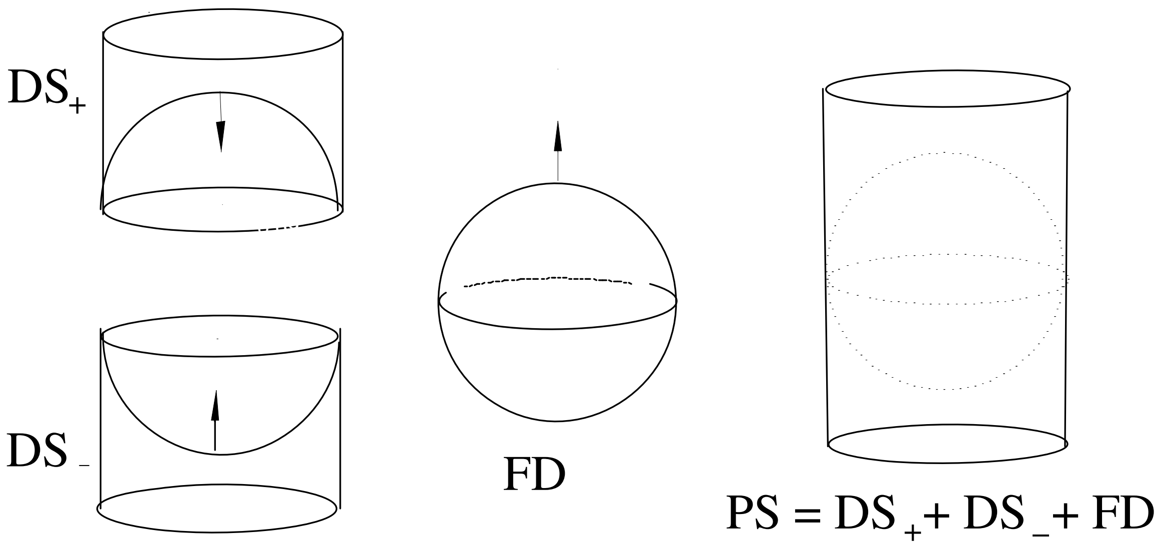
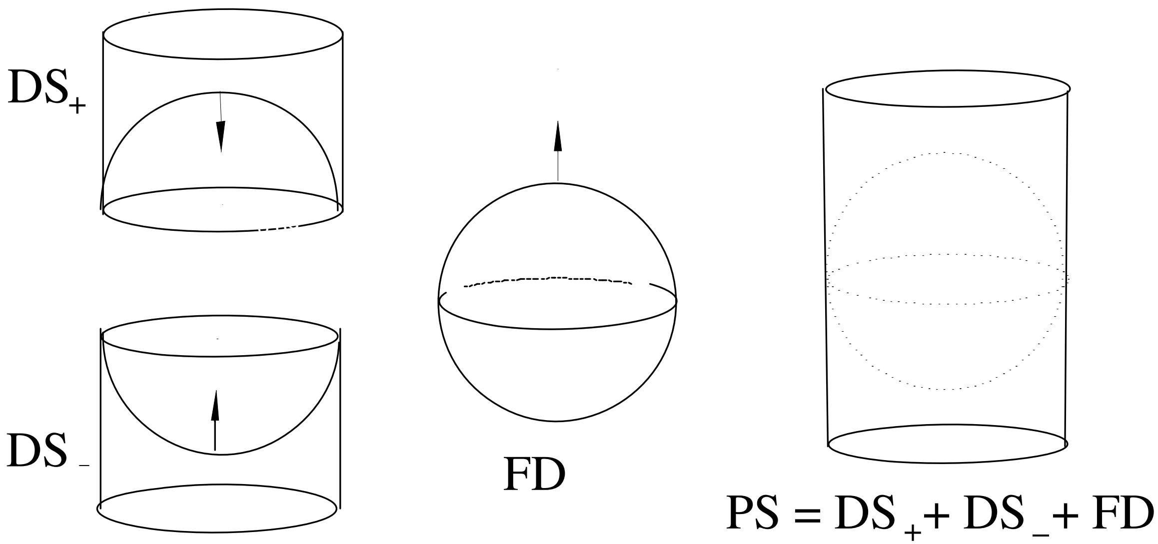




'combinatorial' geometry of representations



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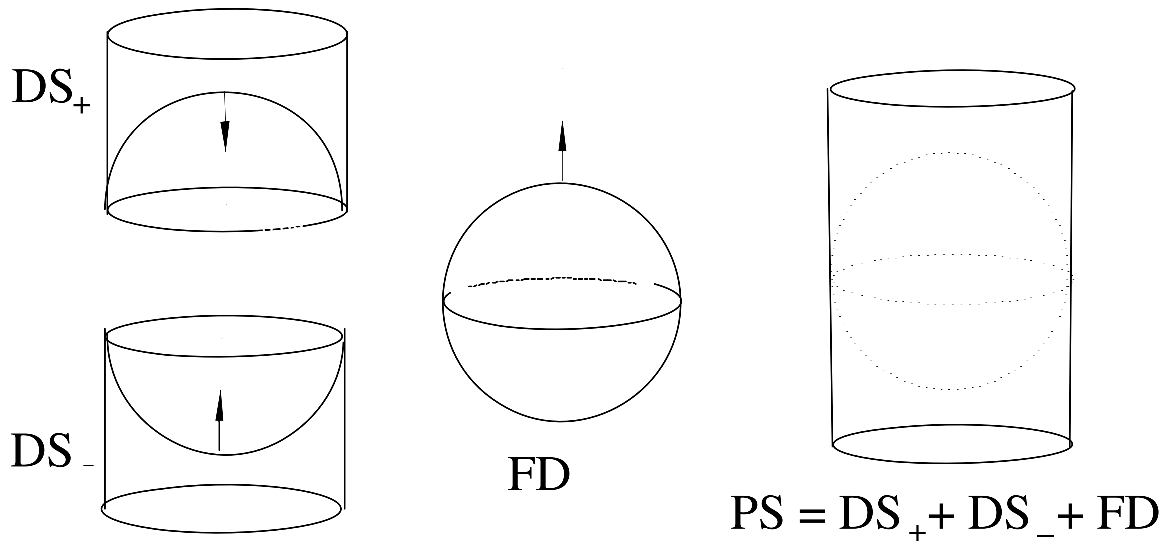

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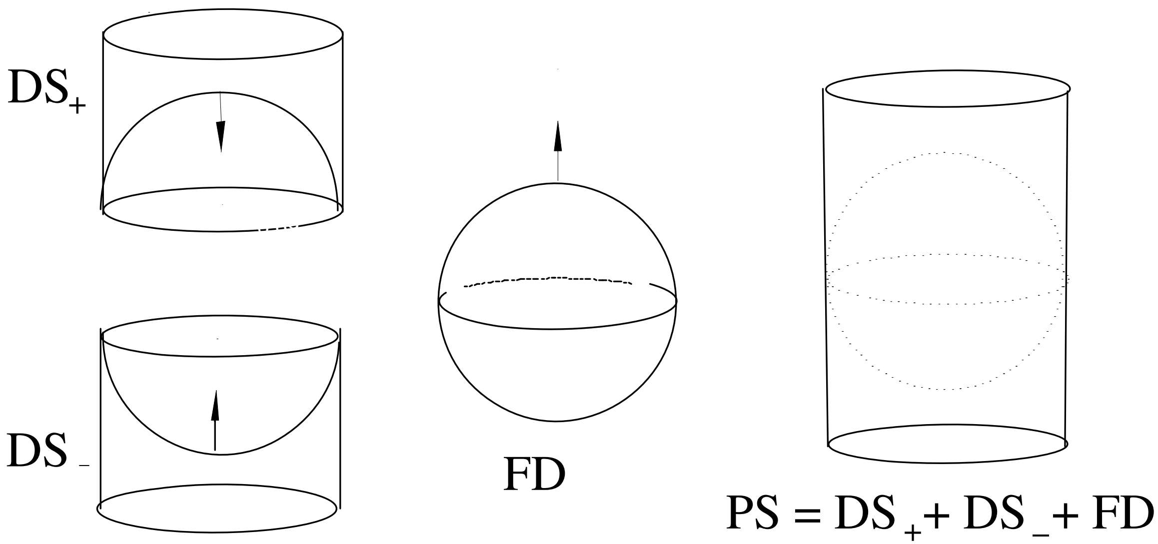
Theorem. (Contour Integral Formula for Characters)

$$\Theta(\Gamma, \lambda, \exp X) = \frac{1}{D(X)} \int_{\xi \in \Gamma} e^{\langle X, \xi \rangle + \sigma_\lambda(d\xi)}$$



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Γ :=contour on complex coadjoint orbit Ω_λ

σ_λ :=complex symplectic 2-from on Ω_λ, $\quad D(X) = \det^{1/2}(\frac{\sinh(\text{ad}X/2)}{\text{ad}X/2})$.

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The wave front set of a representation

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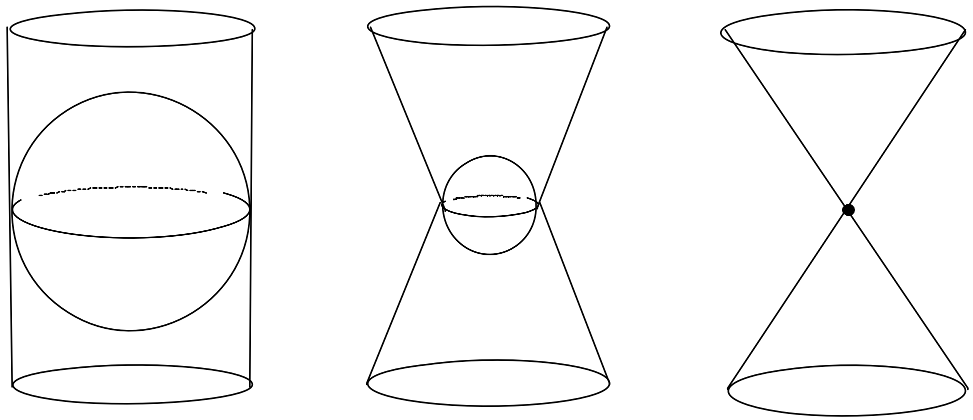

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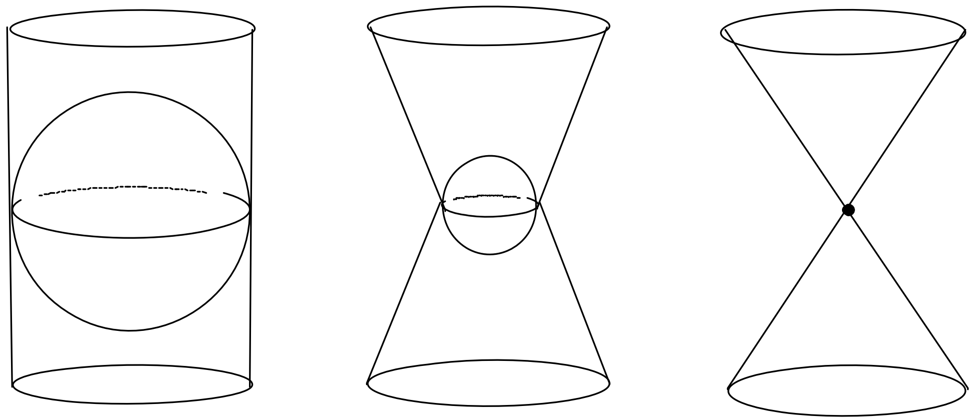

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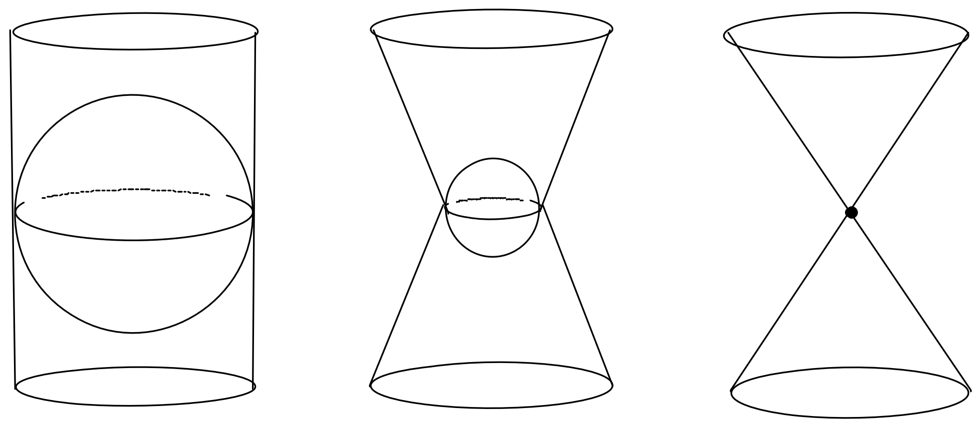

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Theorem. (Asymptotics of characters.) *As* $\lambda \to 0$

$$\int_\Gamma e^{X+\sigma_\lambda} \sim \sum_{\text{nilp. orb.}} c_O(\Gamma,\lambda) \int_O e^{X+\sigma_O}$$

$$\Theta(\Gamma,\lambda,\exp X) \sim \sum c_O(\Gamma,\lambda)\Theta_O$$

Weyl Group Representations

$$W = \{1, s \mid s^2 = 1\}; \quad s\Theta(\lambda) = \Theta(s^{-1}\lambda)$$

sDS$_\pm$ =DS$_\pm$+FD
sFD= $-$FD

(s = reflection along FD)

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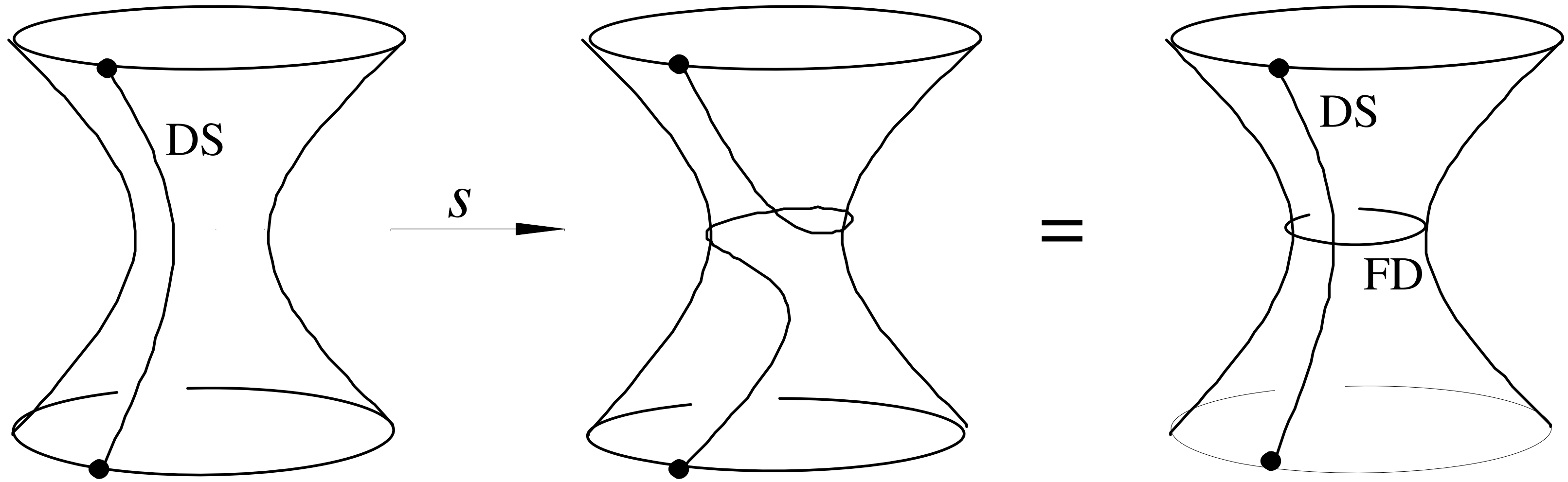
DS
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=
DS
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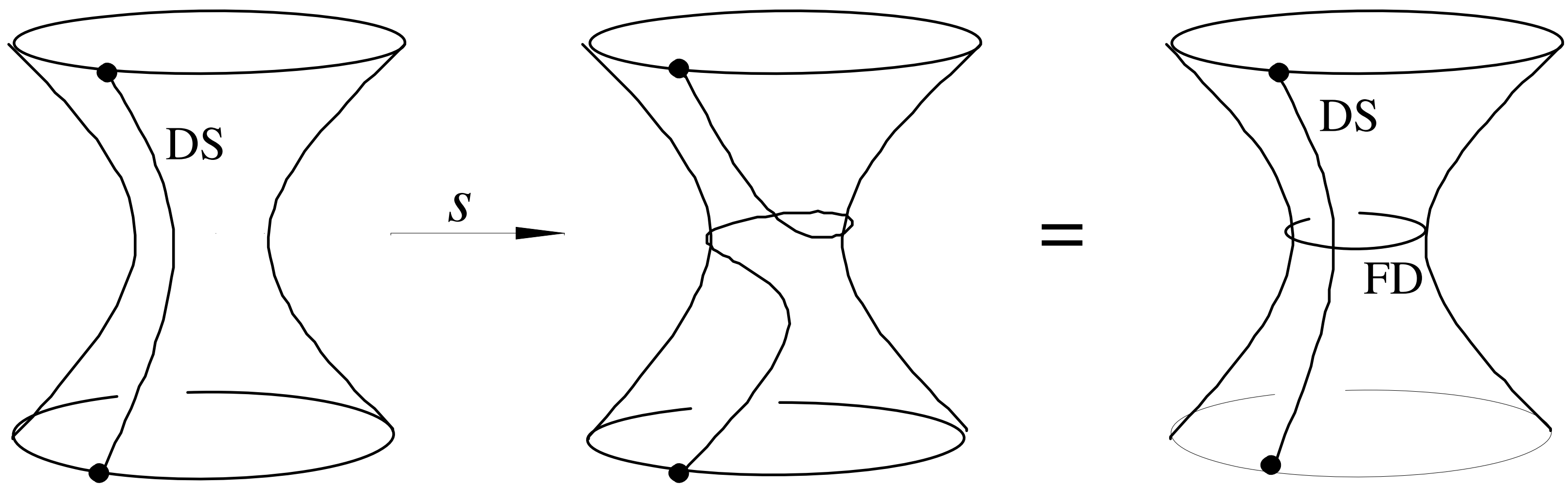
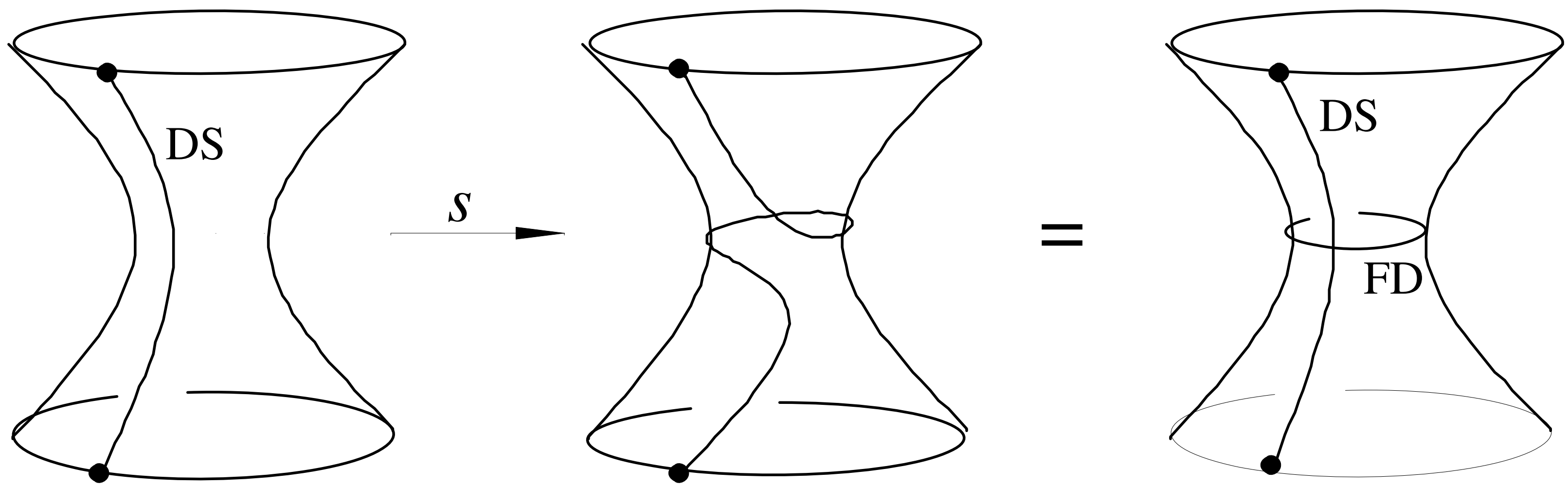


PL deformation of a contour: fixed ends!



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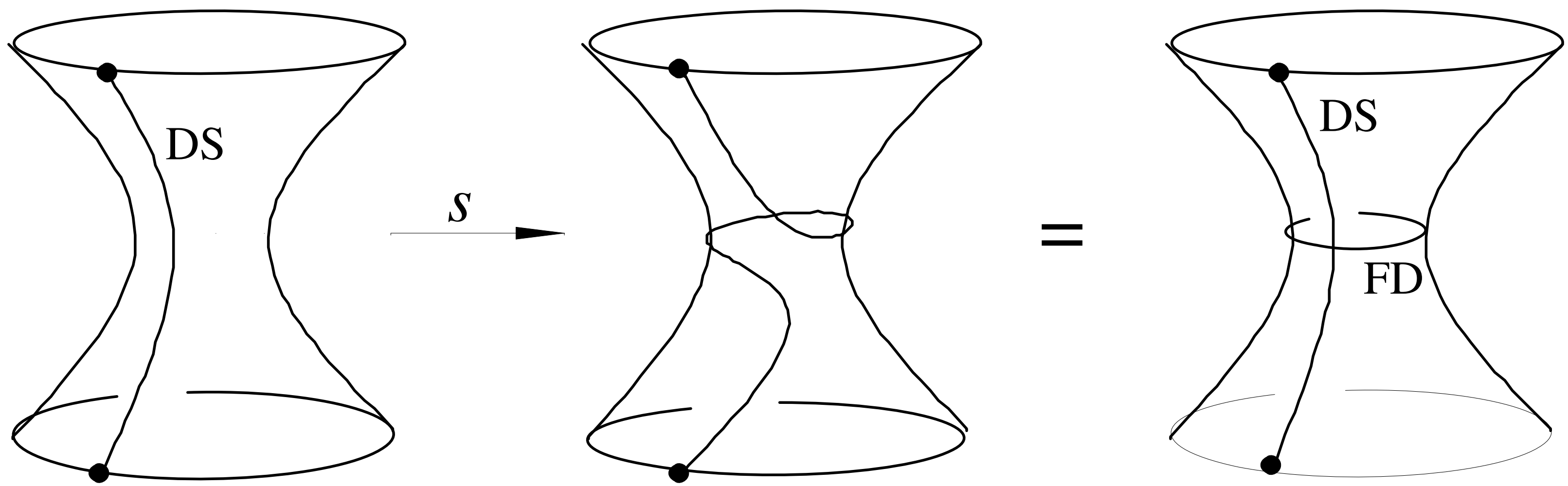

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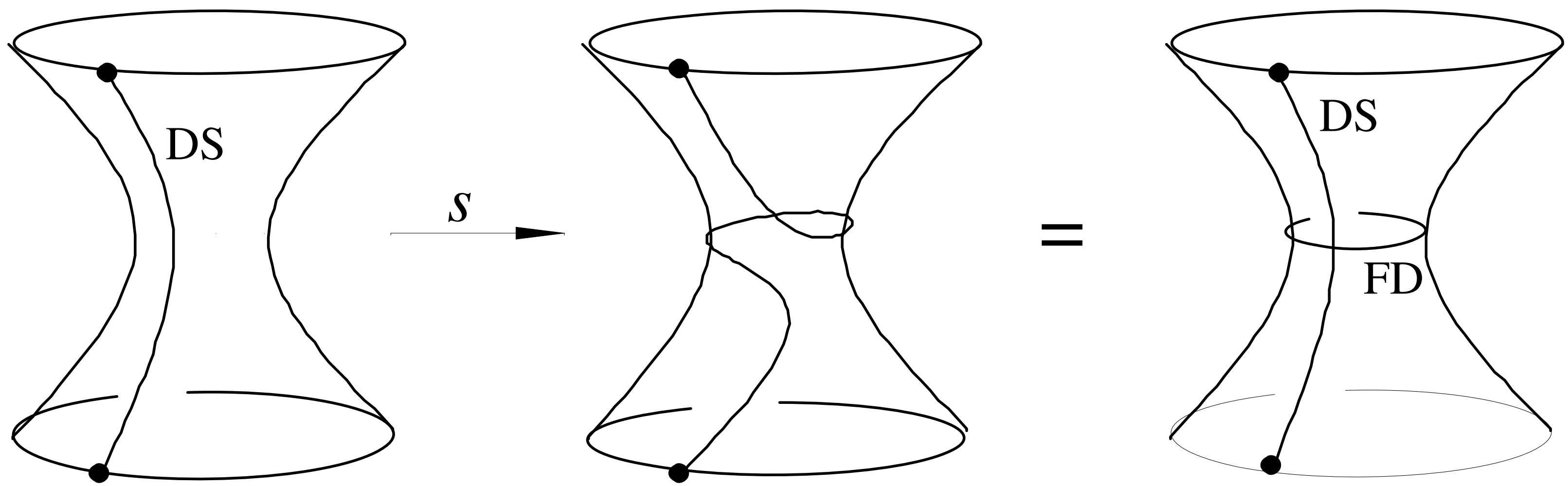

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Flag Manifold

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Example: Gelfand-Tsetlin basis for finite dimensional representations

Harmonic Oscillator

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$\varphi(t) =$orbit of a representation of $\mathbb{R}/2^{-1}\mathbb{Z}$.

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A. Ω_ϕ is rigid in $\mathfrak{sp}\{xp\}^*$ but becomes mobile along with $x = x(\xi, \lambda)$

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Elliptic Contour Integrals

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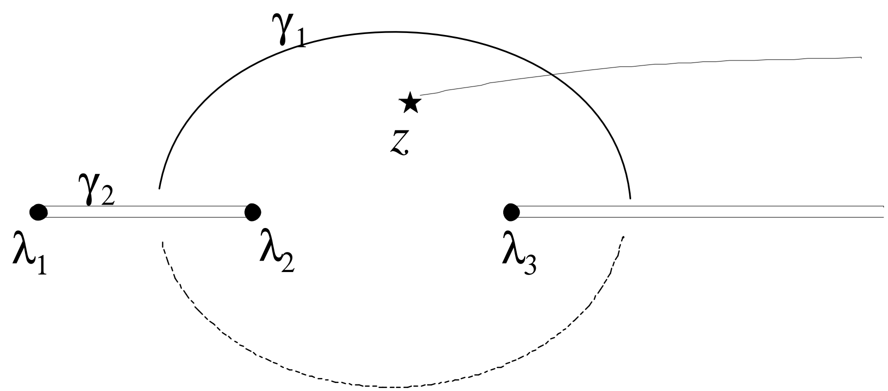

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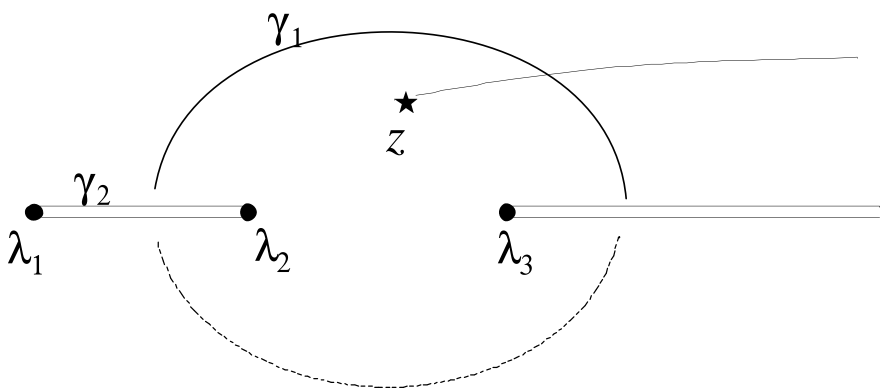

elliptic curve over $\{z\} : E_\lambda = \{w^2 = f(z)\}$; branch cuts $[\lambda_1, \lambda_2], [\lambda_3, \infty]$,
1-homology $\mathrm{H}_1(E_\lambda) = \{\gamma = m_1\gamma_1 + m_2\gamma_2\}$;
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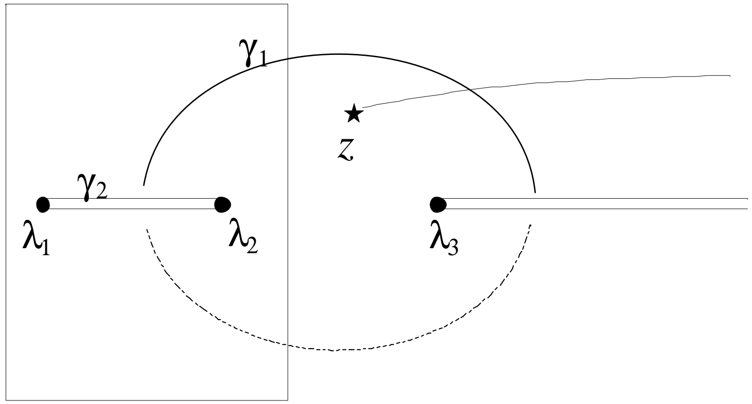

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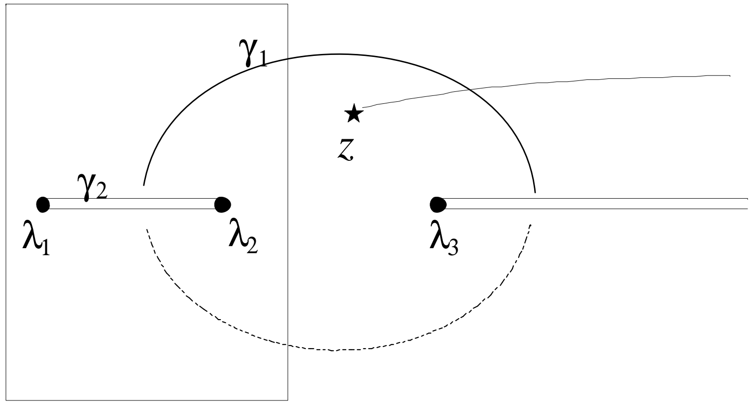

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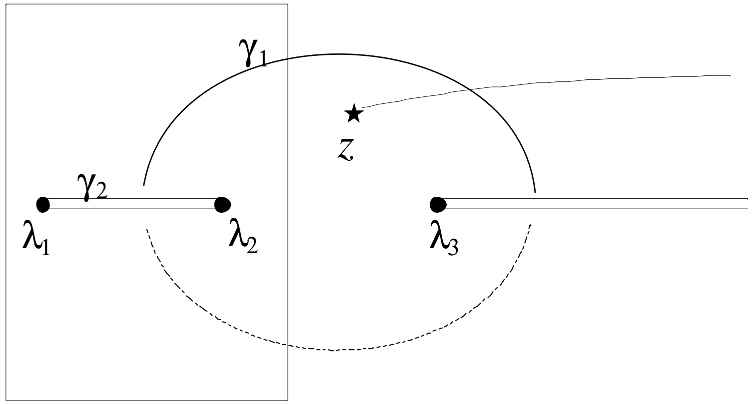

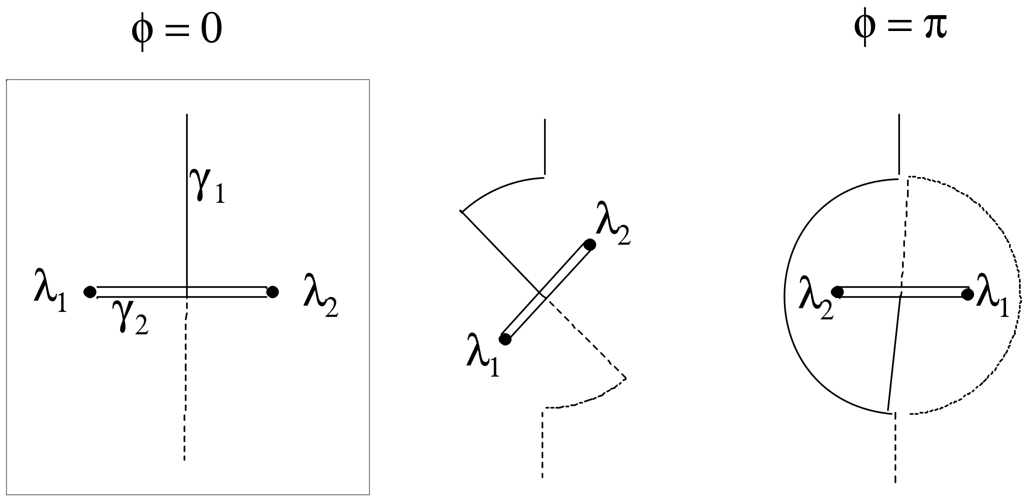

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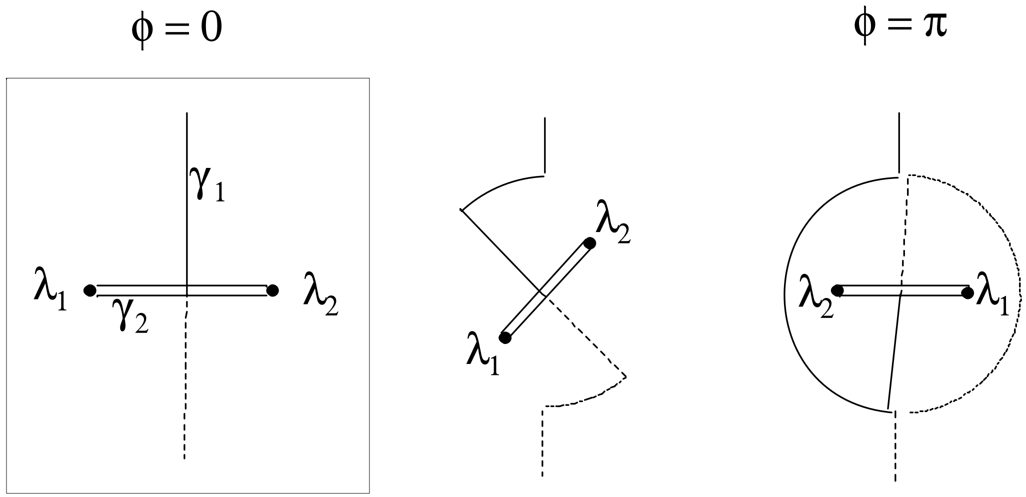

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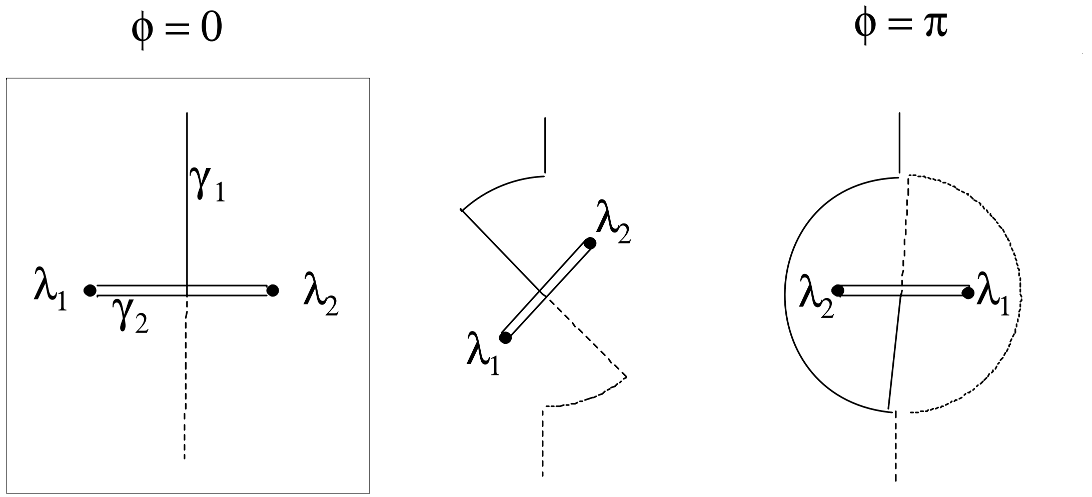

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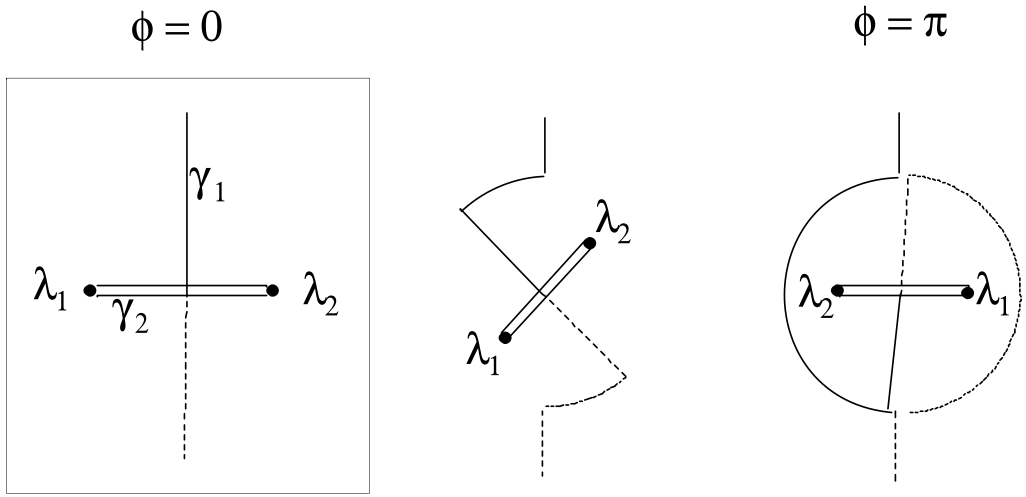

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Transformation Formulas (very old). *On the elliptic curve* $E_\lambda = \{w^2 = f(z)\}$ *the functions* $x_i(\lambda,\xi)$ *split into numerator and denominator in the form*

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(Primeform factorization). The $x_i(\lambda,\xi)$ *transform under* SL$(2,\mathbb{Z}) = \langle S, T\rangle$ *according to the oscillator representation of* SL$(2,\mathbb{Z}/2\mathbb{Z})\times$SL$(2,\mathbb{Z}/2\mathbb{Z})$ *on* $\mathbb{C}^4 = \{X_0,\cdots,X_4\}$.
$>$(to be explained) $>$

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Finite versions $V_2 := \Lambda_2'/\Lambda_2 \approx (\mathbb{Z}/2\mathbb{Z})^2$, Hb$(V_2)$, $\mathcal{H}_2(\Lambda) := \mathcal{H}(V)^{\tilde{\Lambda}_2}$

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$<$

Analogies

Weyl group W	modular group SL$(2,\mathbb{Z})$
complex orbit Ω_λ	phase space $\{(x(\lambda,\xi),p(\lambda,\xi))\}$
contour Γ on Ω_λ	contour γ on $E_\lambda \approx V/\Lambda_\lambda$
orbit integrals for $\Theta(\lambda)$	elliptic integrals for $x_i(\lambda)$
W rep on $'\mathrm{H}(\Omega_\lambda)$	SL$(2,\mathbb{Z})$ rep on $\mathcal{H}(V_2(\lambda))$
Cartan $T_\mathbb{R}$	osc $\mathbb{T} = \{\exp(tH)\})$
rep of $G_\mathbb{R}$	osc rep of Sp(V) on $\{\varphi(x)\}$
?!	toric group on $\{\varphi(\xi)\}$

contours $\int_\Gamma \cdots <$ $\qquad$ $\int_\gamma \cdots <$

deformations $\Gamma = \Gamma(\lambda) <$ $\qquad$ $\gamma = \gamma(\lambda) <$

Notation.

$$M(z) := \prod (z - \xi_i) \quad (i = 1, \cdots, n)$$

$$\frac{1}{M(z)} = \sum \frac{R_i(\xi)}{z - \xi_i}, \quad [R_i(\xi) = \operatorname*{Res}_{z=\xi_i}(\frac{1}{M(z)}) = \frac{1}{M'(\xi_i)}]$$

$$U(z) = \sum_{k=0}^{\infty} \frac{1}{k!} U^{(k)}(0) z^k \text{ holomorphic on } \{z \neq \infty\}$$

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Residue Lemma. *With this notation, the residue relation*

$$\sum_i \frac{1}{2\pi \mathrm{i}} \oint_{(\xi_i)} \frac{U(z)dz}{\prod_j (z - \xi_j)} = -\frac{1}{2\pi \mathrm{i}} \oint_{(\infty)} \frac{U(z)dz}{\prod_j (z - \xi_j)}$$

reads

$$\sum_i U(\xi_i) R_i(\xi) = \sum_l U^{(l)}(\xi_i) S_l(\xi).$$

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Proof. Residue theorem and geometric series. ■

Theorem. *In elliptic coordinates, the oscillator PDE*

$$\sum_{i=1}^{n} \frac{1}{2}(\frac{\partial^2}{\partial x_i^2} + \alpha_i x_i^2)\varphi(x) = E\varphi(x)$$

becomes

$$\sum R_i(\xi) \left[(\sqrt{f(\xi_i)}\frac{\partial}{\partial \xi_i})^2 - A_i(\xi_i) \right] \varphi(\xi) = 0$$

for any functions $A_i(z)$ of the form

$$A_i(z) := \sum_{k=0}^{n-2} a_k z^k + (-\alpha_i \lambda_i + \beta_i) z^{n-1} + (\sum -\alpha_i) z^n$$

with $\sum(-\alpha_i \lambda_i + \beta_i) = E$.

Proof. By calculation based on the Residue Lemma.
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Corollary. *Let $\varphi_1(z), \cdots, \varphi_n(z)$ be n solutions of the ODE $\frac{d^2}{du^2}\varphi_i(z) = A_i(z)\varphi_i(z), \quad [du := dz/\sqrt{f}]$. Then $\varphi(\xi) = \prod \varphi_i(\xi_i)$ is a solution of the oscillator PDE with $E = \sum(-\alpha_i\lambda_i + \beta_i)$.*

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$$H(x,p) = \underset{z=\infty}{\mathit{Res}}\left[\frac{K(x(z),p(z))}{\prod(z-\xi_i)}\right] \Rightarrow H(x,p) = -\sum \underset{z=\xi_i}{\mathit{Res}}\left[\frac{K(x(z),p(z))}{\prod(z-\xi_i)}\right]$$

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Current groups (loop groups, affine groups)

$\mathit{Sp}_z = \mathit{Sp}V_z$, $V_z = \{x_z p_z\}$

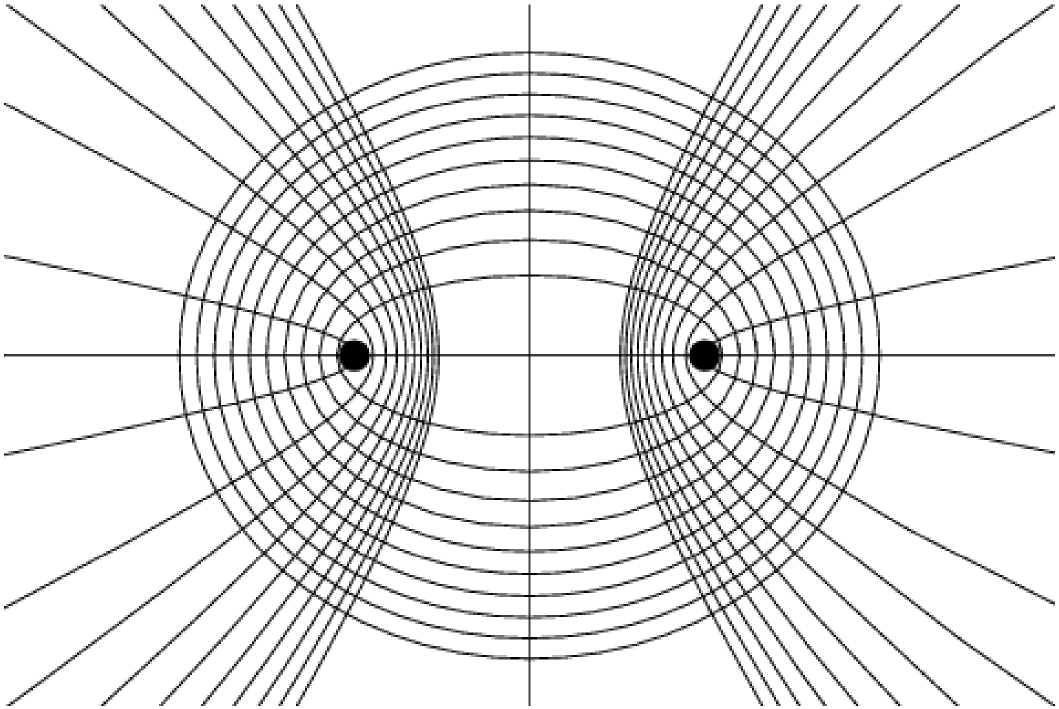
$\Pi_z \mathit{Sp}_z = \{g(z) = \text{arbitrary function}\}$

$\Pi_{z,\mathit{mero}} \mathit{Sp}_z = \{g(z) = \text{meromorphic}\}$

$\Pi_{z,\{\xi_i\}} \mathit{Sp}_z = \{g(z) \text{ has poles only at } z = \xi_i\}$

(Compare: local, global, adelic.)

Picture



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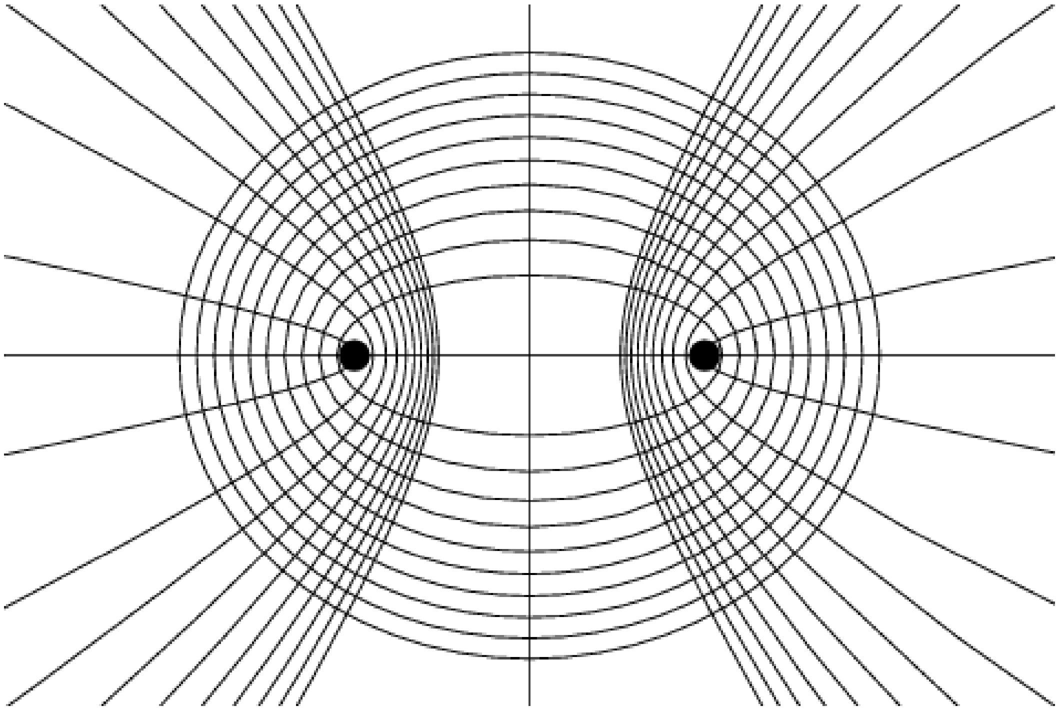

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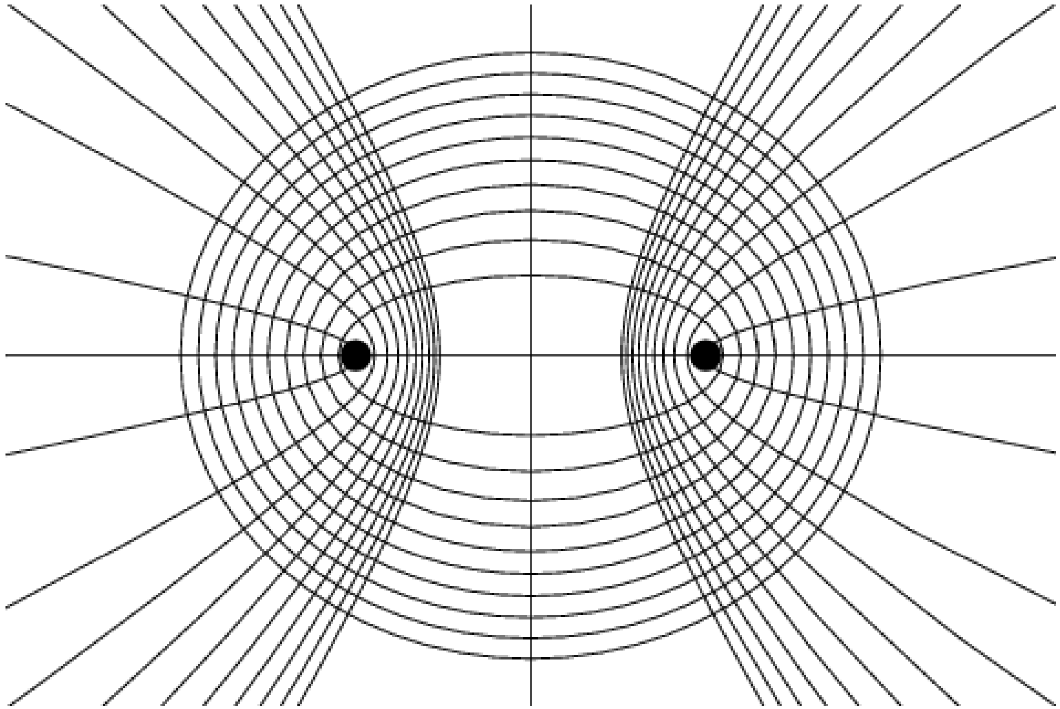

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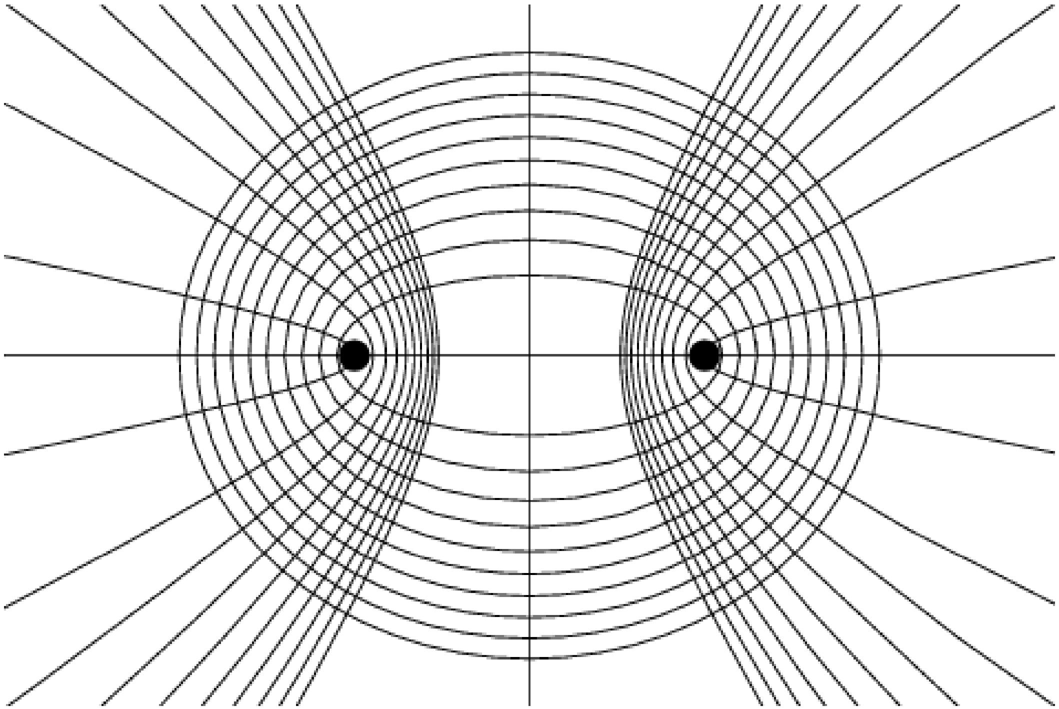

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Compare

$$\frac{(z-\xi_1)(z-\xi_2)}{(z-\lambda_1)(z-\lambda_2)} = 1 - \frac{x_1^2}{z-\lambda_1} - \frac{x_2^2}{z-\lambda_2}$$

with $z = \xi_i(u,\lambda)$, $x = x_i(u,\lambda)$.