Arithmeticity and commensurability of sporadic groups

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Abstract

We prove that the so-called sporadic complex reflection triangle groups in $\mathrm{SU}(2,1)$ are all non-arithmetic but one, and that they are not commensurable to Mostow or Picard lattices (with a small list of exceptions). This provides an infinite list of potential new non-arithmetic lattices in $\mathrm{SU}(2,1)$.

1 Introduction

In [ParPau], Parker and the author considered symmetric triangle groups Δ in $\mathrm{SU}(2,1)$ generated by three complex reflections through angle $2\pi/p$ for $p \geqslant 3$ (the case of order 2 was studied by Parker in [Par1]). By symmetric we mean that the group in question is generated by three complex reflections R_1, R_2 and R_3 with the property that there exists an isometry J of order 3 so that $R_{j+1} = JR_jJ^{-1}$ (where j is taken mod 3). In fact we study the group Γ generated by R_1 and J, which contains Δ with index 1 or 3.

This type of group was first studied by Mostow in [M1] (for $p = 3, 4, 5$), where an additional condition was imposed on the R_j (namely the braid relation $R_iR_jR_i = R_jR_iR_j$); these provided the first examples of non-arithmetic lattices in $\mathrm{SU}(2,1)$. Following that Deligne–Mostow and Mostow constructed further non-arithmetic lattices in $\mathrm{SU}(n,1)$ ($n \leqslant 9$) as monodromy groups of certain hypergeometric functions, in [DM] and [M2] (the lattices from [DM] in dimension 2 were known to Picard who did not consider their arithmetic nature). These lattices are (commensurable with) groups generated by complex reflections R_j with other values of p; see [M2] and [S]. Subsequently no new non-arithmetic lattices have been constructed.

In [ParPau] we showed that symmetric complex reflection triangle groups $\Delta = \langle R_1, R_2, R_3\rangle$, if they are discrete with R_1R_2 and $R_1R_2R_3$ elliptic, come in three flavors: Mostow's lattices, subgroups of Mostow's lattices, and a third class which we called "sporadic groups"(see section 2 for a precise definition). Our main motivation is that these new groups are candidates for non-arithmetic lattices in $\mathrm{SU}(2,1)$. In this paper we analyze the adjoint trace fields $\mathbb{Q}[\mathrm{TrAd}\Gamma]$ of the sporadic groups Γ, and use this to determine which sporadic groups are arithmetic, and which ones are commensurable to Mostow or Picard lattices. The main results are Theorems 4.1 and 5.2, which say in essence that all sporadic groups are non-arithmetic (except one which was studied in [ParPau]), and moreover that they are not commensurable to any of the Mostow or Picard lattices (with an explicit list of possible exceptions).

The only required notions of complex hyperbolic geometry are the definitions of elliptic and regular elliptic isometries, as well as complex reflections. These are standard and can be found for instance in the book [G].

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2 Sporadic groups

In this section we recall the setup and main results from [ParPau]. Our starting point was that groups $\Gamma = \langle R_1, J\rangle$ as defined above can be parametrised up to conjugacy by $\tau = \mathrm{Tr}(R_1 J)$; we denoted $\Gamma(\psi,\tau)$ the group generated by a complex reflection R_1 through angle ψ and a regular elliptic isometry J of order 3 such that $\tau = \mathrm{Tr}(R_1 J)$. The generators for this group were given in the following explicit form:

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{2.1}$$

$$R_1 = \begin{bmatrix} e^{2i\psi/3} & \tau & -e^{i\psi/3}\overline{\tau} \\ 0 & e^{-i\psi/3} & 0 \\ 0 & 0 & e^{-i\psi/3} \end{bmatrix} \tag{2.2}$$

These preserve the Hermitian form $\langle \mathbf{z}, \mathbf{w}\rangle = \mathbf{w}^* H_\tau \mathbf{z}$ where

$$H_\tau = \begin{bmatrix} 2\sin(\psi/2) & -ie^{-i\psi/6}\tau & ie^{i\psi/6}\overline{\tau} \\ ie^{i\psi/6}\overline{\tau} & 2\sin(\psi/2) & -ie^{-i\psi/6}\tau \\ -ie^{-i\psi/6}\tau & ie^{i\psi/6}\overline{\tau} & 2\sin(\psi/2) \end{bmatrix}. \tag{2.3}$$

This always produces a subgroup Γ of $\mathrm{GL}(3,\mathbb{C})$, but the signature of H_τ depends on the values of ψ and τ. We determined the corresponding parameter space for τ for any fixed value of ψ (see sections 2.4 and 2.6 of [ParPau]). When Γ preserves a Hermitian form of signature $(2,1)$ we will say that Γ is *hyperbolic.*

We found necessary conditions for these groups to be discrete, and these conditions produced, along with the groups previously studied by Mostow in [M1], a list of possibly discrete such groups:

Theorem 2.1 *Let R_1 be a complex reflection of order p and J a regular elliptic isometry of order 3 in $\mathrm{PU}(2,1)$. Suppose that R_1J and $R_1R_2 = R_1JR_1J^{-1}$ are elliptic. If the group $\Gamma = \langle R_1, J\rangle$ is discrete then one of the following is true:*

- *Γ is one of Mostow's lattices.*
- *Γ is a subgroup of one of Mostow's lattices.*
- *Γ is one of the sporadic groups listed below.*

Mostow's lattices correspond to $\tau = e^{i\phi}$ for some angle ϕ; subgroups of Mostow's lattices to $\tau = e^{2i\phi} + e^{-i\phi}$ for some angle ϕ, and sporadic groups (this can be taken as a definition) are those for which τ takes one of the 18 values $\{\sigma_1, \overline{\sigma_1}, ..., \sigma_9, \overline{\sigma_9}\}$ where the σ_i are given in the following list:

$$\begin{array}{ccc} \sigma_1 := e^{i\pi/3} + e^{-i\pi/6}\,2\cos(\pi/4) & \sigma_2 := e^{i\pi/3} + e^{-i\pi/6}\,2\cos(\pi/5) & \sigma_3 := e^{i\pi/3} + e^{-i\pi/6}\,2\cos(2\pi/5) \\ \sigma_4 := e^{2\pi i/7} + e^{4\pi i/7} + e^{8\pi i/7} & \sigma_5 := e^{2\pi i/9} + e^{-i\pi/9}\,2\cos(2\pi/5) & \sigma_6 := e^{2\pi i/9} + e^{-\pi i/9}\,2\cos(4\pi/5) \\ \sigma_7 := e^{2\pi i/9} + e^{-i\pi/9}\,2\cos(2\pi/7) & \sigma_8 := e^{2\pi i/9} + e^{-i\pi/9}\,2\cos(4\pi/7) & \sigma_9 := e^{2\pi i/9} + e^{-i\pi/9}\,2\cos(6\pi/7). \end{array}$$

Therefore, for each value of $p \geqslant 3$, we have a finite number of new groups to study, the $\Gamma(2\pi/p, \sigma_i)$ and $\Gamma(2\pi/p, \overline{\sigma_i})$ which are hyperbolic. We determined exactly which sporadic groups are hyperbolic (see table in section 3.3 of [ParPau]); notably these exist for all values of p, and more precisely:

Proposition 2.1 *For $p \geqslant 4$ and $\tau = \sigma_1, \sigma_2, \sigma_3, \overline{\sigma_4}, \sigma_5, \sigma_6, \sigma_7, \overline{\sigma_8}$ or σ_9, $\Gamma(2\pi/p, \tau)$ is hyperbolic.*

When we study the question of arithmeticity of these groups, we will use the list of all hyperbolic sporadic groups, as well as the following normalization of the entries of our matrices (proposition 2.8 of [ParPau]):

Proposition 2.2 *The maps R_1, R_2 and R_3 may be conjugated within $\mathrm{SU}(2,1)$ and scaled so that their matrix entries lie in the ring $\mathbb{Z}[\tau, \overline{\tau}, e^{\pm i\psi}]$.*

Explicitly, we conjugate the previous matrices by $C = \mathrm{diag}(e^{-i\psi/3},\, 1,\, e^{i\psi/3})$ and rescale by $e^{-i\psi/3}$. Conjugating by C and rescaling by $2\sin(\psi/2)$ also brings H_τ to a Hermitian matrix with entries in the same ring $R = \mathbb{Z}[\tau, \overline{\tau}, e^{\pm i\psi}]$. Therefore, a hyperbolic $\Gamma(\psi, \tau)$ can be realized as a subgroup of $\mathrm{SU}(H, R)$ where H is an $R-$defined Hermitian form of signature $(2, 1)$.

Finally, we showed that some of the hyperbolic sporadic groups are non-discrete (see Corollary 4.2, Proposition 4.5 and Corollary 6.4 of [ParPau]):

Proposition 2.3 *For $p \geqslant 3$ and (τ or $\overline{\tau} = \sigma_3, \sigma_8$ or σ_9), $\Gamma(2\pi/p, \tau)$ is not discrete. Also, for $p \geqslant 3$, $p \neq 5$ and (τ or $\overline{\tau} = \sigma_6$), $\Gamma(2\pi/p, \tau)$ is not discrete.*

3 Trace fields

The trace field $\mathbb{Q}\,[\mathrm{Tr}\Gamma]$ is a classical invariant for a finitely generated subgroup Γ of a linear group G. It is invariant under conjugacy, but not commensurability. (We will say that two subgroups Γ_1 and Γ_2 of G are *commensurable* if there exists $g \in G$ such that $\Gamma_1 \cap g\Gamma_2 g^{-1}$ has finite index in both Γ_1 and $g\Gamma_2 g^{-1}$). To obtain a commensurability invariant for such Γ, one can consider one of the following fields. Either the trace field $\mathbb{Q}\left[\mathrm{Tr}\Gamma^{(n)}\right]$ (where $\Gamma^{(n)}$ is the subgroup of Γ generated by n-th powers, for $\Gamma \subset \mathrm{GL}(n, \mathbb{C})$), as in [MR] for $\mathrm{SL}(2, \mathbb{C})$ or [Mc] for $\mathrm{SU}(2, 1)$. Another possibility is the adjoint trace field $\mathbb{Q}\,[\mathrm{TrAd}\Gamma]$, given by the adjoint representation: $\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g})$, as in [M1], [M2] or [DM] for $\mathrm{SU}(n, 1)$. The following result can be found for instance in [DM] (Proposition 12.2.1):

Proposition 3.1 $\mathbb{Q}\,[\mathrm{TrAd}\Gamma]$ *is a commensurability invariant.*

This is the field that we will use here, as it is more convenient for our purposes. Indeed, this invariant trace field has been computed for all known non-arithmetic lattices in $\mathrm{SU}(2, 1)$ (see lists on p. 251 of [M1] and p. 86 of [DM]), and moreover it is easy to compute (or at least estimate) by the following result:

Proposition 3.2 *For $\gamma \in \mathrm{SU}(2, 1)$, $\mathrm{TrAd}(\gamma) = |\mathrm{Tr}(\gamma)|^2$.*

This statement is used several times in [M1], where it is referred to as lemma 4.2, but unfortunately doesn't appear in the final edition.

Proof. If U is a unitary group (of any signature), the adjoint representation of U is isomorphic to the representation $U \otimes \overline{U}$. □

We use this to find the following bounds for $\mathbb{Q}\,[\mathrm{TrAd}\Gamma(\psi, \tau)]$:

Proposition 3.3 $\mathbb{Q}\left[\cos\psi, |\tau|^2, \mathrm{Re}\tau^3, \mathrm{Re}(e^{-i\psi}\tau^3)\right] \subset \mathbb{Q}\,[\mathrm{TrAd}\Gamma(\psi, \tau)] \subset \mathbb{Q}\left[\tau, \overline{\tau}, e^{i\psi}\right] \cap \mathbb{R}$.

Proof. The second inclusion follows from Propositions 2.2 and 3.2. For the first inclusion, we use Proposition 3.2 and compute $|\mathrm{Tr}(\gamma)|^2$ for various words γ, using the table of traces from section 4.1 of [ParPau] (see also formulae in [Pr]):

- $|\mathrm{Tr}R_1|^2 = 5 + 4\cos\psi$
- $|\mathrm{Tr}R_1 J|^2 = |\tau|^2$ (by definition of τ)
- $|\mathrm{Tr}(R_1 J)^2|^2 = |\tau|^4 + 4|\tau|^2 - 4\mathrm{Re}\tau^3$
- $|\mathrm{Tr}(J^{-1}R_1)^2|^2 = |\tau|^4 + 4|\tau|^2 - 4\mathrm{Re}(e^{-i\psi}\tau^3)$ □

We list the corresponding elements of $\mathbb{Q}\,[\mathrm{TrAd}\Gamma(2\pi/p, \sigma_i)]$ in the following table. Numbers in the last three columns are not the values of $|\tau|^2$, Re τ^3 or Re $(e^{-i\psi}\tau^3)$, but rather new algebraic numbers added to $\mathbb{Q}\,[\mathrm{TrAd}\Gamma(2\pi/p, \sigma_i)]$ by these values. For example, the first four zeroes in the fourth column indicate that the corresponding Re τ^3 is already in $\mathbb{Q}\left[\cos\psi, |\tau|^2\right]$.

	$\cos\psi$	$\|\tau\|^2$	Re τ^3	Re $(e^{-i\psi}\tau^3)$
σ_1	$\cos 2\pi/p$	0	0	$\sqrt{2}\sin 2\pi/p$
σ_2	$\cos 2\pi/p$	$\cos\pi/5$	0	$\sin 2\pi/p$
σ_3	$\cos 2\pi/p$	$\cos 3\pi/5$	0	$\sin 2\pi/p$
σ_4	$\cos 2\pi/p$	0	0	$\sqrt{7}\sin 2\pi/p$
σ_5	$\cos 2\pi/p$	0	$\cos 2\pi/5$	$\sqrt{3}\sin 2\pi/p$
σ_6	$\cos 2\pi/p$	0	$\cos 4\pi/5$	$\sqrt{3}\sin 2\pi/p$
σ_7	$\cos 2\pi/p$	$\cos\pi/7$	0	$\sqrt{3}\sin 2\pi/p$

4 Arithmeticity

In [ParPau] (Propositions 6.5 and 6.6) we proved that only one of the sporadic groups with $p = 3$, namely $\Gamma(2\pi/3, \overline{\sigma_4})$, is contained in an arithmetic lattice in $\mathrm{SU}(2,1)$. In this section we extend this to higher values of p, and show that in fact this group is the only such example among all sporadic groups:

Theorem 4.1 *For $p \geqslant 3$ and $\tau \in \{\sigma_1, \overline{\sigma_1}, ..., \sigma_9, \overline{\sigma_9}\}$, $\Gamma(2\pi/p, \tau)$ is contained in an arithmetic lattice in $\mathrm{SU}(2,1)$ if and only if $p = 3$ and $\tau = \overline{\sigma_4}$.*

We will use the following criterion for arithmeticity:

Proposition 4.1 *Let E be a purely imaginary quadratic extension of a totally real field F, and H an E-defined Hermitian form of signature (2,1) such that a sporadic group Γ is contained in $\mathrm{SU}(H; \mathcal{O}_E)$. Then Γ is contained in an arithmetic lattice in $\mathrm{SU}(2,1)$ if and only if for all $\varphi \in \mathrm{Gal}(F)$ not inducing the identity on $\mathbb{Q}[\mathrm{TrAd}\Gamma]$, the form ${}^{\varphi}H$ is definite.*

This follows from lemma 4.1 of [M1]. Hypotheses (1) and (3) of that lemma (that $\mathbb{Q}[\mathrm{TrAd}\Gamma]$ is a totally real field, and $\mathrm{TrAd}\gamma$ is an algebraic integer for all $\gamma \in \Gamma$ respectively) are verified by Propositions 2.2 and 3.2, using the special values of τ for sporadic groups.

We will prove theorem 4.1 in several parts using this criterion. The first result follows the same lines as the corresponding one in [ParPau]:

Proposition 4.2 *The sporadic group $\Gamma(2\pi/p, \tau)$ is not contained in an arithmetic lattice in $\mathrm{SU}(2,1)$, with the following possible exceptions:*

- *$\tau = \sigma_1$ and ($p = 4$ or $p \geqslant 8$)*
- *$\tau = \sigma_2$ and (3 or 4 or 5 divides p)*
- *$\tau = \overline{\sigma_2}$ and ($p = 8, 9, 10, 12, 14, 15, 16$ or 18)*
- *$\tau = \overline{\sigma_4}$ and ($p = 3$ or $p \geqslant 7$)*
- *$\tau = \sigma_5$ and 5 divides p*
- *$\tau = \sigma_7$ and 7 divides p.*

Proof. We conjugate the generators and Hermitian form as in Proposition 2.2 so that their entries lie in the ring $\mathbb{Z}[\tau, \overline{\tau}, e^{\pm i\psi}]$, and are therefore algebraic integers in the field $\mathbb{Q}[\tau, \overline{\tau}, e^{i\psi}]$. (Recall that in our cases $\psi = 2\pi/p$). We then find in each case a number field E as in Proposition 4.1, containing $\mathbb{Q}[\tau, \overline{\tau}, e^{i\psi}]$, and a Galois conjugation of E which acts nontrivially on $\mathbb{Q}[\mathrm{TrAd}\Gamma]$ and sends the Hermitian form to another indefinite form. For the values of τ and p which are not excluded above, we can use the same argument as in [ParPau], namely that one of the Galois conjugations of E sends the parameter τ to another value for which we know that the Hermitian form is indefinite (from our description of the parameter space). This requires using a Galois conjugation fixing $e^{2i\pi/p}$. The details are as follows:

- For $\tau = \sigma_1$ or $\overline{\sigma_1}$, let $E = \mathbb{Q}[e^{i\pi/6}, e^{i\pi/4}, e^{2i\pi/p}]$. If p is not divisible by 3 or 4, the Galois conjugation sending $e^{i\pi/6}$ to $e^{-i\pi/6}$, $e^{i\pi/4}$ to $e^{-i\pi/4}$, and fixing $e^{2i\pi/p}$ sends σ_1 to $\overline{\sigma_1}$. The corresponding Hermitian form is indefinite for $p = 3,4,5,6,7$. This works for $p = 5$ or 7, but for $p = 3$, 4 or 6 we need to find another Galois conjugation. For $p = 3$ or 6, sending $e^{i\pi/6}$ to $e^{7i\pi/6}$ (and for compatibility $e^{i\pi/4}$ to $e^{-i\pi/4}$) fixes $e^{2i\pi/3}$ (respectively $e^{2i\pi/6}$) and sends σ_1 to $e^{4i\pi/3}\overline{\sigma_1}$, which is equivalent to $\overline{\sigma_1}$. These various Galois conjugations act nontrivially on $\mathrm{Re}(e^{-i\psi}\tau^3) = 5\cos\psi + 5\sqrt{2}\sin\psi$, which is in $\mathbb{Q}\,[\mathrm{Tr}\mathrm{Ad}\Gamma]$.

- For $\tau \in \{\sigma_2, \overline{\sigma_2}, \sigma_3, \overline{\sigma_3}\}$, let $E = \mathbb{Q}[e^{i\pi/6}, e^{i\pi/5}, e^{2i\pi/p}]$. If p is not divisible by 3 or 4 or 5, the Galois conjugation sending $e^{i\pi/5}$ to $e^{3i\pi/5}$, $e^{i\pi/6}$ to $e^{7i\pi/6}$ and fixing $e^{2i\pi/p}$ swaps σ_2 and σ_3, as well as $\overline{\sigma_2}$ and $\overline{\sigma_3}$. The Hermitian form corresponding to σ_2 and σ_3 is indefinite for all $p \geqslant 3$; for $\overline{\sigma_2}$ it is indefinite for $3 \leqslant p \leqslant 19$, and for $\overline{\sigma_3}$ it is indefinite for $3 \leqslant p \leqslant 6$. This Galois conjugation acts nontrivially on $|\tau|^2 = 2 + 2\cos(\pi/5)$ (respectively $2 + 2\cos(2\pi/5)$), which is in $\mathbb{Q}\,[\mathrm{Tr}\mathrm{Ad}\Gamma]$.

  If p is not divisible by 2 or 3, the Galois conjugation sending $e^{i\pi/6}$ to $e^{-i\pi/6}$ and fixing the 2 other generators of E sends σ_2 to $\overline{\sigma_2}$. This works unless $p = 8, 9, 10, 12, 14, 15, 16, 18$.

- For $\tau = \sigma_4$ or $\overline{\sigma_4}$, let $E = \mathbb{Q}[e^{2i\pi/7}, e^{2i\pi/p}]$, which contains $i\sqrt{7} = \sigma_4 - \overline{\sigma_4}$. If p is not divisible by 7, the Galois conjugation sending $e^{2i\pi/7}$ to $e^{-2i\pi/7}$ and fixing the other generator of E sends σ_4 to $\overline{\sigma_4}$. The corresponding Hermitian form is indefinite for $p = 4, 5, 6$. This Galois conjugation acts nontrivially on $8\mathrm{Re}(e^{-i\psi}\tau^3) = 20\cos\psi + 4\sqrt{7}\sin\psi$, which is in $\mathbb{Q}\,[\mathrm{Tr}\mathrm{Ad}\Gamma]$.

- For $\tau \in \{\sigma_5, \overline{\sigma_5}, \sigma_6, \overline{\sigma_6}\}$, let $E = \mathbb{Q}[e^{i\pi/9}, e^{2i\pi/5}, e^{2i\pi/p}]$. If p is not divisible by 5, the Galois conjugation sending $e^{2i\pi/5}$ to $e^{4i\pi/5}$ and fixing the 2 other generators of E sends σ_5 to σ_6, and $\overline{\sigma_5}$ to $\overline{\sigma_6}$. The Hermitian form corresponding to σ_5 and σ_6 is indefinite for all $p \geqslant 3$; for $\overline{\sigma_5}$ it is indefinite for $p = 2, 4$, and for $\overline{\sigma_6}$ it is indefinite for $4 \leqslant p \leqslant 29$. This Galois conjugation acts nontrivially on $\mathrm{Re}\tau^3 = 11/2 + 11\cos(2\pi/5)$ (respectively $11/2 + 11\cos(4\pi/5)$), which is in $\mathbb{Q}\,[\mathrm{Tr}\mathrm{Ad}\Gamma]$.

  If p is not divisible by 3, the Galois conjugation sending $e^{i\pi/9}$ to $e^{-i\pi/9}$ and fixing the 2 other generators of E sends σ_6 to $\overline{\sigma_6}$. This works for $p = 5$ (the only case where Proposition 2.3 doesn't tell us that $\Gamma(2\pi/p, \sigma_6)$ and $\Gamma(2\pi/p, \overline{\sigma_6})$ are nondiscrete).

- For $\tau \in \{\sigma_7, \overline{\sigma_7}, \sigma_8, \overline{\sigma_8}, \sigma_9, \overline{\sigma_9}\}$, let $E = \mathbb{Q}[e^{i\pi/9}, e^{2i\pi/7}, e^{2i\pi/p}]$. If p is not divisible by 7, the Galois conjugation sending $e^{2i\pi/7}$ to $e^{6i\pi/7}$ and fixing the 2 other generators of E sends σ_7 to σ_9 and σ_9 to σ_8, and $\overline{\sigma_7}$ to $\overline{\sigma_9}$ and $\overline{\sigma_9}$ to $\overline{\sigma_8}$. The Hermitian form corresponding to σ_7, $\overline{\sigma_8}$ and σ_9 is indefinite for all $p \geqslant 4$ (even 3 for σ_7, σ_9); for $\overline{\sigma_7}$ it is indefinite for $p = 2$, for σ_8 it is indefinite for $4 \leqslant p \leqslant 41$, and for $\overline{\sigma_9}$ it is indefinite for $4 \leqslant p \leqslant 8$. This Galois conjugation acts nontrivially on $|\tau|^4 + |\tau|^2 - 2\mathrm{Re}\tau^3 = 3 + 2\cos(2\pi/7)$ (respectively $3 + 2\cos(4\pi/7)$ and $3 + 2\cos(6\pi/7)$), which is in $\mathbb{Q}\,[\mathrm{Tr}\mathrm{Ad}\Gamma]$.

- Finally, we know from Proposition 2.3 that, for $\tau \in \{\sigma_3, \overline{\sigma_3}, \sigma_8, \overline{\sigma_8}, \sigma_9, \overline{\sigma_9}\}$, $\Gamma(2\pi/p, \tau)$ is nondiscrete for all p, and in particular is not contained in an arithmetic lattice in $\mathrm{SU}(2,1)$. $\square$

We then examine the remaining cases, where we must now take into account the effect of our various Galois conjugations on $\psi = e^{2i\pi/p}$. We will use the following notation. In all cases, the number field E is a cyclotomic field $\mathbb{Q}\left[e^{2i\pi/r}\right]$; the Galois group of E consists of the automorphisms φ_n sending $e^{2i\pi/r}$ to $e^{2in\pi/r}$, for $(n,r) = 1$. We will use the following criterion (Corollary 2.7 of [ParPau]), which expresses the determinant κ of the Hermitian matrix H_τ in a convenient way:

Lemma 4.1 *When $\tau = e^{i\alpha} + e^{i\beta} + e^{-i\alpha - i\beta}$ and $\sin(\psi/2) \geqslant 0$, the matrix H_τ has signature $(2,1)$ if and only if*

$$\kappa = 8\sin(3\alpha/2 + \psi/2)\sin(3\beta/2 + \psi/2)\sin\bigl(-3(\alpha+\beta)/2 + \psi/2\bigr) < 0. \tag{4.1}$$

Proposition 4.3 *$\Gamma(2\pi/p, \tau)$ is not contained in an arithmetic lattice in $\mathrm{SU}(2,1)$ for:*

- *$\tau = \sigma_1$ and ($p = 4$ or $p \geqslant 8$)*
- *$\tau = \sigma_2$ and 3 or 4 or 5 divides p*
- *$\tau = \overline{\sigma_4}$ and $p \geqslant 7$*

- $\tau = \sigma_5$ *and* 5 *divides* p
- $\tau = \sigma_7$ *and* 7 *divides* p.

Proof. The argument is the following. In each case we find a Galois conjugation φ (acting nontrivially on $\mathbb{Q}[\mathrm{TrAd}\Gamma]$) such that two of $\varphi(e^{3i\alpha/2})$, $\varphi(e^{3i\beta/2})$, and $\varphi(e^{-3i(\alpha+\beta)/2})$ lie in the open upper half of the unit circle, and the third in the open lower half (or, in the case of $\tau = \overline{\sigma_4}$, all three in the lower half). Then this property is stable, i.e. if $\varphi(\psi)$ is small enough, adding $\varphi(\psi)/2$ to each of the three angles will not change it (where we think of φ as acting on angles). We now give more details:

- As previously, for $\tau = \sigma_1$ let $E = \mathbb{Q}[e^{i\pi/6}, e^{i\pi/5}, e^{2i\pi/p}]$; we will use $\varphi \in \mathrm{Gal}(E)$ fixing σ_1 up to a cube root of unity. With the notation of Lemma 4.1, the corresponding triple $(3\alpha/2, 3\beta/2, -3(\alpha+\beta)/2)$ is $(\pi/2, \pi/8, -5\pi/8)$. We can achieve $\varphi_n(\sigma_1) = \sigma_1$ (up to a cube root of unity) by sending $e^{i\pi/4}$ to $e^{\pm i\pi/4}$ and fixing $e^{i\pi/6}$ (up to a cube root of unity), or by sending $e^{i\pi/4}$ to $e^{\pm 3i\pi/4}$ and $e^{i\pi/6}$ to $e^{7i\pi/6} = -e^{i\pi/6}$ (up to a cube root of unity). This means that n is congruent to (1 or -1 mod 8) and (1 or 5 or 9 mod 12) in the first case, and to (3 or -3 mod 8) and (3 or 7 or 11 mod 12) in the second. We win if we can find such an n, coprime with p and such that $n\pi/p < \pi/2$, i.e. $n \leqslant 2p+1$ (this is the largest angle by which one can rotate the 3 points on the unit circle without any of them changing sides). The first few solutions to the above congruences are $n = (1), 3, 9, 11, 17, 19, 25, 27, 33, 35, 41$. Start with $n = 3$; this works as long as 3 doesn't divide p and $p \geqslant 7$. We check that for $p = 4$, $\varphi_5(\kappa) < 0$ (and $\varphi_5(\sqrt{2}) \neq \sqrt{2}$). Assume then that 3 divides p, and use $n = 11$; this works as long as 11 doesn't divide p and $p \geqslant 23$. This leaves $p = 9, 12, 15, 18, 21$; we check that $n = 5$ works for $p = 9, 18, 21$, $n = 7$ works for $p = 12$, and $n = 11$ for $p = 15$. Assume then that 33 divides p, and use $n = 17$; this works as long as 17 doesn't divide p and $p \geqslant 34$. This leaves $p = 33$, where we check that $\varphi_5(\kappa) < 0$. We then go on in this fashion (skipping solutions like 27 and 33 which are divisible by 3), assuming that $3 \times 11 \times 17$ divides p and using $n = 19$ and so on. In this fashion p increases multiplicatively, whereas solutions to the above congruences increase additively, therefore such n exist by a wider and wider margin. We conclude inductively that such an n exists for p large enough (and we have checked the few exceptions for small p).

- As previously, for $\tau = \sigma_2$ or σ_3 let $E = \mathbb{Q}[e^{i\pi/6}, e^{i\pi/5}, e^{2i\pi/p}]$ and consider $\varphi \in \mathrm{Gal}(E)$ sending $e^{i\pi/5}$ to $e^{3i\pi/5}$ and $e^{i\pi/6}$ to $e^{7i\pi/6} = -e^{i\pi/6}$. Then φ swaps σ_2 and σ_3. With the notation of Lemma 4.1, the corresponding triples $(3\alpha/2, 3\beta/2, -3(\alpha+\beta)/2)$ are $(\pi/2, \pi/20, -11\pi/20)$ when $\tau = \sigma_2$, and $(\pi/2, 7\pi/20, -17\pi/20)$ when $\tau = \sigma_3$. Now when 3 or 4 or 5 divide p, φ also acts on $e^{2i\pi/p}$.

  If 4 divides p, writing $p = 4k$, $(e^{2i\pi/p})^k = i = (e^{i\pi/6})^3$ is sent to $-i$, so $\varphi(e^{2i\pi/p})$ must be a k-th root of $-i$, in other words $\omega_k.e^{-i\pi/2k}$ for a k-th root of unity ω_k. In fact, if 3 or 5 don't divide p one can send $e^{2i\pi/p}$ to any $\omega_k.e^{-i\pi/2k}$, say with $\omega_k = e^{2i\pi/k}$ (this gives a better bound on p than 1). Then $\psi/2$ is sent to $3\psi/2$ (because $-\pi/2k + 2\pi/k = 3\pi/2k$), and the argument works for $3\pi/p < 11\pi/20$ ($p \geqslant 6$) when $\tau = \sigma_3$, and $3\pi/p < 17\pi/20$ ($p \geqslant 4$) when $\tau = \sigma_2$. There remain the cases where 5 divides p, as well as $\tau = \sigma_3$ and $p = 4$. In the latter case one can check that $\varphi_{13}(\kappa) < 0$, with $\varphi_{13}(\cos 3\pi/5) \neq \cos 3\pi/5$.

  Now suppose that 5 divides p but 3 or 4 do not, and write $p = 5k$. As above, one can send $e^{2i\pi/p}$ to $e^{6i\pi/p}$, and the same argument tells us that $\varphi(\kappa) < 0$ for $p \geqslant 4$ when $\tau = \sigma_2$ and $p \geqslant 6$ when $\tau = \sigma_3$. When $p = 5$ and $\tau = \sigma_3$, one can again check that $\varphi_{13}(\kappa) < 0$ (with $\varphi_{13}(\cos 3\pi/5) \neq \cos 3\pi/5$).

  If 3 divides p, we find $\varphi \in \mathrm{Gal}(E)$ as above, namely we require that $\varphi(e^{i\pi/5}) = e^{3i\pi/5}$ or $e^{-3i\pi/5}$ and $\varphi(e^{i\pi/6}) = e^{7i\pi/6}$ up to a cube root of unity, so that φ swaps σ_2 and σ_3 (up to a cube root of unity). Such a φ is realized as a φ_n if (and only if) n is congruent to (3 or -3 mod 10) and (3 or 7 or 11 mod 12). The values of such n are 3, 7, 23, 27, 43, 47... Moreover, with the angle triples as above, $\varphi_n(\kappa) < 0$ for $n\pi/p < 17\pi/20$ (when $\tau = \sigma_2$) or $n\pi/p < \pi/2$ (when $\tau = \sigma_3$). We may use $n = 7$ as long as 7 doesn't divide p, which works for $p \geqslant 9$ when $\tau = \sigma_2$, and $p \geqslant 15$ when $\tau = \sigma_3$. We then check the cases $p = 6$ and $\tau = \sigma_2$, as well as $p = 6, 9, 12$ and $\tau = \sigma_3$. It turns out that $n = 7$ works for all of these (renormalizing, when $p = 6$, $7 \times 2\pi/6$ as $2\pi/6$). Now if 7 also divides p, we use the next solution $n = 23$, which works for $p \geqslant 47$ when $\tau = \sigma_2$, and $p \geqslant 28$ when $\tau = \sigma_3$, as long as 23 doesn't divide p. We check that $n = 11$ works for $p = 21$ ($\tau = \sigma_2, \sigma_3$) and $p = 42$ ($\tau = \sigma_2$). One can then assume that 21×23 divides p, and so on. We conclude inductively as above.

- For $\tau = \overline{\sigma_4}$, E is as previously $\mathbb{Q}[e^{2i\pi/7}, e^{2i\pi/p}]$, and $(3\alpha/2, 3\beta/2, -3(\alpha+\beta)/2) = (-3\pi/7, -6\pi/7, 9\pi/7)$. If 7 doesn't divide p, consider $\varphi \in \mathrm{Gal}(E)$ fixing $e^{2i\pi/7}$ and sending $e^{2i\pi/p}$ to $e^{2in\pi/p}$ with $(n,p) = 1$ and $1 < n \leqslant 3p/7$ (this is possible as $p \geqslant 7$). Then $n\pi/p \leqslant 3\pi/7$ as required.

  If 7 divides p, say $p = 7k$, one can again fix $e^{2i\pi/7}$ and send $e^{2i\pi/p}$ to a k-th root of itself; when $k \geqslant 3$, letting $\varphi(e^{2i\pi/p}) = e^{2i\pi(1/k+1/p)}$ works (i.e. $\varphi(\kappa) < 0$), because $\pi/k + \pi/p < 3\pi/7$. Remain only the cases $p = 7$, where one can check that $\varphi_2(\kappa) < 0$ (with $\varphi_2(\cos 2\pi/7) \neq \cos 2\pi/7$), and $p = 14$ where one can check that $\varphi_9(\kappa) < 0$ (with $\varphi_9(\cos\pi/7) \neq \cos\pi/7$).

- As previously, for $\tau = \sigma_5$ or σ_6 let $E = \mathbb{Q}[e^{i\pi/9}, e^{2i\pi/5}, e^{2i\pi/p}]$ and consider $\varphi \in \mathrm{Gal}(E)$ sending $e^{2i\pi/5}$ to $e^{4i\pi/5}$ and fixing $e^{i\pi/9}$. Then φ swaps σ_5 and σ_6. With the notation of Lemma 4.1, the corresponding triples $(3\alpha/2, 3\beta/2, -3(\alpha+\beta)/2)$ are $(\pi/3, 13\pi/30, -23\pi/30)$ when $\varphi(\tau) = \sigma_5$, and $(\pi/3, 31\pi/30, -41\pi/30)$ when $\varphi(\tau) = \sigma_6$.

  If 5 divides p, say $p = 5k$, φ must send $e^{2i\pi/p}$ to a k-th root of $e^{4i\pi/5}$, and one can choose any of these if 3 does not divide p, such as $e^{4i\pi/5k}$. When $\tau = \sigma_5$ this works for $2\pi/p \leqslant 17\pi/30$ ($p \geqslant 4$), and when $\tau = \sigma_6$ for $2\pi/p \leqslant 11\pi/30$ ($p \geqslant 6$). When $p = 5$ and $\tau = \sigma_6$, one can check that $\varphi_4(\kappa) < 0$ (with $\varphi_4(\sqrt{3}\sin(2\pi/5)) \neq \sqrt{3}\sin(2\pi/5))$.

  Now if 3 also divides p, we must look more closely at how φ is defined above. Namely such a φ is a φ_n if and only if n is congruent to 2 mod 5 and 1 mod 18. The smallest such n is 37. However one can relax slightly the definition of φ to allow $\varphi(e^{i\pi/9}) = \omega_3 e^{i\pi/9}$ for any cubic root of unity ω_3 (this does not affect τ). The conditions are then that n should be congruent to (2 mod 5) and (1 or 7 or 13 mod 18). We can then use $n = 7$, unless 7 divides p. In that case φ_7 would work for $7\pi/p \leqslant 17\pi/30$ ($p \geqslant 13$) when $\tau = \sigma_6$, and for $7\pi/p \leqslant 11\pi/30$ ($p \geqslant 20$) when $\tau = \sigma_5$. Since at this point 15 divides p, there remains only the case where $p = 15$ and $\tau = \sigma_6$, in which case one can check that $\varphi_{11}(\kappa) < 0$ (with $\varphi_{11}(\cos(2\pi/15)) \neq \cos(2\pi/15)$.

  Finally, if 7 also divides p (at this point p is divisible by 105), we can do the same thing. That is, we claim that there exists n congruent to (2 mod 5) and (1 or 7 or 13 mod 18), coprime with p and such that $n\pi/p \leqslant 11\pi/30$ (i.e. $n \leqslant 11p/30$). For $p = 105k$, $n = 37$ satisfies these conditions for $1 \leqslant k \leqslant 36$. After that, suppose that 37 divides p and so on; we conclude inductively as above.

- As previously, for $\tau = \sigma_7$ let $E = \mathbb{Q}[e^{i\pi/9}, e^{2i\pi/7}, e^{2i\pi/p}]$ and consider $\varphi_n \in \mathrm{Gal}(E)$ sending $e^{2i\pi/7}$ to $e^{6i\pi/7}$ (resp. $e^{-2i\pi/7}$) and fixing $e^{i\pi/9}$ (up to a cube root of unity) This means that n should be congruent to (3 resp. -1 mod 7) and (1 or 7 or 13 mod 18). Then $\varphi_n(\sigma_7) = \sigma_9$ (resp.σ_7), and the corresponding triple $(3\alpha/2, 3\beta/2, -3(\alpha+\beta)/2)$ is $(\pi/3, 47\pi/42, -61\pi/42)$ (resp. $(\pi/3, 11\pi/42, -25\pi/42)$). With these values, $\varphi_n(\kappa) < 0$ when $n\pi/p \leqslant 19\pi/42$ (resp. $n\pi/p \leqslant 25\pi/42$). The smallest such n is 13, which works for $p \geqslant 22$ (as long as 13 doesn't divide p). It remains to check $p = 7, 14$ or 21 (7 is assumed to divide p): $n = 5$ works when $p = 7$ or 21, and $n = 11$ works when $p = 14$. If 13 divides p, use the next solution $n = 31$, and so on. We conclude inductively as above.

$\square$

Lemma 4.2 *For $\tau = \overline{\sigma_2}$ and $p = 8, 9, 10, 12, 14, 15, 16, 18$, $\Gamma(2\pi/p, \tau)$ is not contained in an arithmetic lattice in* $\mathrm{SU}(2,1)$.

Proof. For each of these values we find a Galois conjugation φ_n of E such that $\varphi_n(\kappa) < 0$, where $\kappa = \det H_\tau$, and acting nontrivially on $\mathbb{Q}[\mathrm{TrAd}\Gamma]$. For this last condition, it suffices to check that $\varphi_n(\cos 2\pi/p) \neq \cos 2\pi/p$ (this is true for all cases below, except $n = 7$ and $p = 8$, in which case $\varphi_7(\cos\pi/5) \neq \cos\pi/5$). The condition $\varphi_n(\kappa) < 0$ can easily be checked, for instance numerically. We claim that the following φ_n satisfy these conditions when $\tau = \overline{\sigma_2}$: φ_7 works for $p = 8, 9, 10, 12$, and φ_{11} works for $p = 14, 15, 16, 18$. $\square$

5 Commensurability

In this section we compare the adjoint trace fields of our sporadic groups with those of the previously known lattices in $\mathrm{SU}(2,1)$, namely the Picard and Mostow lattices (see [DM], [M1], [M2], [S], [T] and [Par2] for an overview). From the lists on p. 251 of [M1], p. 86 of [DM] and 548-549 of [T] we see that for these lattices Γ, $\mathbb{Q}[\mathrm{Tr}\mathrm{Ad}\Gamma]$ is always of the form $\mathbb{Q}[\cos 2\pi/d]$, where:

- $d = 3,4,5,6,8,9,10,12,18$ for the arithmetic Picard lattices
- $d = 12,15,20,24$ for the nonarithmetic Picard lattices
- $d = 1,8,10,12,15,18$ for the arithmetic Mostow lattices
- $d = 12,15,18,20,24,30,42$ for the nonarithmetic Mostow lattices.

Moreover, only two nonarithmetic noncocompact lattices are known in $\mathrm{SU}(2,1)$, both with $d = 12$. We take this opportunity to make the following remark:

Remark 5.1 The nonarithmetic Picard and Mostow lattices in $\mathrm{SU}(2,1)$ fall into at least 7 and at most 9 distinct commensurability classes.

Indeed there are 6 distinct adjoint trace fields ($d = 15$ and 30 give the same field), and for $d = 12$ there are two classes, one cocompact and the other noncocompact. Also, there are a priori 15 examples, but Mostow ([M2]) and Sauter ([S]) find commensurabilities among some of them. See [Par2] for more details.

Now we use the values from Proposition 3.3 to distinguish commensurability classes of sporadic groups, from each other and from the Picard and Mostow lattices. We will also use the fact that arithmeticity and cocompactness are commensurability invariants. We summarize the results from this section in the following statement:

Theorem 5.2 *For $p \geqslant 3$ and $\tau \in \{\sigma_1, \overline{\sigma_1}, ..., \sigma_9, \overline{\sigma_9}\}$, the sporadic groups $\Gamma(2\pi/p, \tau)$ are not commensurable to any Picard or Mostow lattice, except possibly when:*

- $p = 4$ *or* 6 *(and τ is any sporadic value)*
- $p = 3$ *and* $\tau = \sigma_7$
- $p = 5$ *and* τ *or* $\overline{\tau} = \sigma_1, \sigma_2$
- $p = 7$ *and* $\tau = \overline{\sigma_4}$
- $p = 8$ *and* $\tau = \sigma_1$
- $p = 10$ *and* $\tau = \sigma_1, \sigma_2, \overline{\sigma_2}$
- $p = 12$ *and* $\tau = \sigma_1, \sigma_7$
- $p = 20$ *and* $\tau = \sigma_1, \sigma_2$
- $p = 24$ *and* $\tau = \sigma_1$

The first observation follows simply from the order of the complex reflections in the group, in other words the fact that $\mathbb{Q}[\mathrm{Tr}\mathrm{Ad}\Gamma(2\pi/p,\tau)]$ contains $\cos 2\pi/p$. The values of $p \geqslant 3$ that we rule out are the divisors of $12,15,18,20,24,30,42$.

Lemma 5.1 *For $p \neq 3,4,5,6,7,8,9,10,12,14,15,18,20,21,24,30,42$, the sporadic groups $\Gamma(2\pi/p,\tau)$ are not commensurable to any Picard or Mostow lattice. Moreover, they fall into infinitely many distinct commensurability classes.*

We then examine the remaining values of p, where we can rule out most cases except when $p = 3,4$ or 6:

Lemma 5.2 *When $p \in \{5,7,8,9,10,12,14,15,18,20,21,24,30,42\}$, the sporadic groups $\Gamma(2\pi/p,\tau)$ are not commensurable to any Picard or Mostow lattice, except possibly when:*

- $p=5$ *and* τ *or* $\overline{\tau}=\sigma_1,\sigma_2$
- $p=7$ *and* $\tau=\overline{\sigma_4}$
- $p=8$ *and* $\tau=\sigma_1$
- $p=10$ *and* $\tau=\sigma_1,\sigma_2,\overline{\sigma_2}$
- $p=12$ *and* $\tau=\sigma_1,\sigma_7$
- $p=20$ *and* $\tau=\sigma_1,\sigma_2$
- $p=24$ *and* $\tau=\sigma_1$

Proof. We use the values found for $\mathbb{Q}[\mathrm{Tr}\mathrm{Ad}\Gamma]$ in section 3, listed in the table at the end of that section, as well as the following criterion.

Let $p \geqslant 3$, $p \neq 6$ and $d \in \mathbb{N}$. Then: $\sin 2\pi/p = \cos(p-4)\pi/2p$ is in $\mathbb{Q}[\cos 2\pi/d]$ if and only if:

- p divides d (if 4 divides p)
- $2p$ divides d (if p is even but not divisible by 4)
- $4p$ divides d (if p is odd).

This allows us to rule out the following cases:

- $p=7,9,14,15,18,21,30,42$ when τ or $\overline{\tau}=\sigma_1$
- $p=7,8,9,12,14,15,18,21,24,30,42$ when τ or $\overline{\tau}=\sigma_2$ or σ_3
- $p=5,8,9,10,12,14,15,18,20,21,24,30$ when τ or $\overline{\tau}=\sigma_4$
- $p=5,7,8,9,10,12,14,15,18,20,21,24,30,42$ when τ or $\overline{\tau}=\sigma_5$ or σ_6
- $p=5,7,8,9,10,14,15,18,20,21,24,30,42$ when τ or $\overline{\tau}=\sigma_7$. □

Lemma 5.3 $\Gamma(2\pi/3,\overline{\sigma_4})$ *is not commensurable to any Picard or Mostow lattice.*

Proof. Recall that this is the only arithmetic sporadic group. $\mathbb{Q}[\mathrm{Tr}\mathrm{Ad}\Gamma(2\pi/3,\overline{\sigma_4})]$ contains $\sqrt{21}$, which is not in $\mathbb{Q}[\cos 2\pi/d]$ for $d=1,3,4,5,6,8,9,10,12,15,18$. □

Lemma 5.4 $\Gamma(2\pi/3,\sigma_1)$, $\Gamma(2\pi/3,\overline{\sigma_1})$, $\Gamma(2\pi/3,\sigma_5)$, $\Gamma(2\pi/5,\sigma_3)$ *and* $\Gamma(2\pi/5,\overline{\sigma_3})$ *are not commensurable to any Picard or Mostow lattice.*

Proof. Indeed, in $\Gamma(2\pi/3,\sigma_1)$, $\Gamma(2\pi/3,\overline{\sigma_1})$, $\Gamma(2\pi/5,\sigma_3)$ and $\Gamma(2\pi/5,\overline{\sigma_3})$, R_1R_2 is parabolic (see [ParPau]), and in $\Gamma(2\pi/3,\sigma_5)$ $R_2(R_1J)^5$ is parabolic (details to appear in a forthcoming paper). It follows from Godement's compactness criterion that such a group cannot be commensurable to a cocompact lattice. Therefore it suffices to check that these groups are not commensurable to the noncocompact Picard and Mostow lattices, which both have adjoint trace field equal to $\mathbb{Q}[\cos 2\pi/12]$. Now for $\tau=\sigma_1$ or $\overline{\sigma_1}$, $\mathbb{Q}[\mathrm{Tr}\mathrm{Ad}\Gamma(2\pi/3,\tau)]$ contains $\sqrt{2}\sin 2\pi/p = \sqrt{6}/2$ which is not in $\mathbb{Q}[\cos 2\pi/12]$, and in the three other cases $\mathbb{Q}[\mathrm{Tr}\mathrm{Ad}\Gamma]$ contains $\cos 2\pi/5$ which is not in $\mathbb{Q}[\cos 2\pi/12]$ either. □

Lemma 5.5 $\Gamma(2\pi/3,\overline{\sigma_1})$, $\Gamma(2\pi/3,\sigma_2)$ *and* $\Gamma(2\pi/3,\overline{\sigma_2})$ *are not discrete, and therefore not commensurable to any Picard or Mostow lattice.*

Proof. In the first of these groups $R_1(R_1J)^4$ is elliptic of infinite order, and in the two others $R_1(R_1J)^5$ is elliptic of infinite order (details to appear in a forthcoming paper). □

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