Name KEY **7:30 Midterm Exam 3 – Ver 2**

1. **(ch4)** Complete enough of the following to add to 100%.

(a) [100%] Let S be the vector space of all continuous functions defined on $-2 \le x \le 2$. Define V to be the set of all functions $f(x)$ in S such that $\int_0^2 x f(x)dx = 0$. Prove that V is a subspace of S, by using the Subspace Criterion.

(b) [30%] Let S be the set of all 3×1 column vectors $\mathbf{x}$ with components x_1, x_2, x_3. Assume the usual $\mathcal{R}^3$ rules for addition and scalar multiplication. Let V be the subset of S defined by the dot product equations $\mathbf{a} \cdot \mathbf{x} = \mathbf{0}$, $\mathbf{b} \cdot \mathbf{x} = \mathbf{0}$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors in S. Prove that V is a subspace of S.

(c) [70%] Solve for the unknowns a, b, c, d in the system of equations below by augmented matrix RREF methods, showing all details. Briefly, show the entire snapshot sequence to the **rref**, then display the general solution with variables t_1, t_2, ...

$$\begin{array}{ccccccccc} a & + & b & - & 2c & + & d & = & 1 \\ & + & b & + & 2c & + & & = & 0 \\ a & + & 2b & + & & + & d & = & 1 \\ a & + & 3b & + & 2c & + & d & = & 1 \end{array}$$

(a) Zero is in V because $\int_0^2 0\,dx = 0$. If f, g are in V, then $\int_0^2 (c_1 f + c_2 g)\, x\, dx = c_1 (\int_0^2 x f\, dx) + c_2 (\int_0^2 x g\, dx) = c_1(0) + c_2(0) = 0$. By the subspace criterion, V is a subspace of S.

(b) Let matrix A have rows $\vec{a}, \vec{b}, \vec{0}$, that is, $A = \text{aug}(\vec{a}, \vec{b}, \vec{0})^T$. Then the dot product equations are exactly $A\vec{x} = \vec{0}$. By Theorem 2 of E&P, V is a subspace of S.

(c)
$$\left(\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 1 & 1 \\ 1 & 3 & 2 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 \end{array}\right) \begin{array}{l} \\ \\ \text{combo} \\ \text{combo} \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \\ \\ \text{combo} \\ \text{combo} \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -4 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \text{combo} \\ \\ \\ \end{array}$$

rref found

$$\begin{cases} \boxed{a} - 4c + d = 1 \\ \boxed{b} + 2c = 0 \end{cases}$$

gen Sol
$$\begin{cases} a = 4t_1 - t_2 + 1 \\ b = -2t_1 \\ c = t_1 \\ d = t_2 \end{cases}$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = t_1 \begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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2. **(ch5)** Complete (a) and either (b) or (c). Do not do both (b) and (c).

(a) [30%] Given $8x''(t)+16x'(t)+2x(t) = 0$, which represents a damped spring-mass system with $m = 8$, $c = 16$, $k = 2$, **solve** the differential equation [20%] and **classify** the answer as over-damped, critically damped or under-damped [10%].

(b) [70%] Display by variation of parameters a particular solution x_p for the equation $x'' + 2x' = f(t)$. Leave the answer in unevaluated integral form. Evaluate all symbols except $f(t)$ appearing in (33) of Edwards-Penney.

(c) [70%] Find by undetermined coefficients the steady-state periodic solution for the equation $x'' + 2x' + 2x = 5\sin(t)$.

(a)

$$4r^2 + 8r + 1 = 0$$

$$r = -\frac{8}{8} \pm \frac{\sqrt{64-16}}{8}$$

$$= -1 \pm \frac{1}{2}\sqrt{3}$$

$$= r_1, r_2$$

$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
over-damped
$r_1 = -1 + \sqrt{3}/2$, $r_2 = -1 - \sqrt{3}/2$

(b)

$$r^2 + 2r = 0$$

$$r(r+2) = 0$$

$$r = 0, -2$$

$$x_1 = e^{0t} = 1$$

$$x_2 = e^{-2t}$$

$$W = \begin{vmatrix} 1 & e^{-2t} \\ 0 & -2e^{-2t} \end{vmatrix} = -2e^{-2t}$$

$$(33)\quad x_p = \left(\int x_2 \frac{(-f)}{W}\right) x_1 + \left(\int x_1 \frac{f}{W}\right) x_2$$

$$x_p = \left(\int \frac{f(t)}{2}\,dt\right) + \left(\int -\frac{f(t)}{2} e^{2t}\,dt\right) e^{-2t}$$

(c)

$$x = d_1 \cos t + d_2 \sin t$$

$$x' = -d_1 \sin t + d_2 \cos t$$

$$x'' = -d_1 \cos t - d_2 \sin t$$

$$x'' + 2x' + 2x = 5 \sin t$$

$$2x' + x = 5 \sin t \qquad \text{because } x'' + x = 0$$

$$-2d_1 \sin t + 2d_2 \cos t + d_1 \cos t + d_2 \sin t = 5 \sin t$$

$$\begin{cases} -2d_1 + d_2 = 5 \\ 2d_2 + d_1 = 0 \end{cases} \qquad \text{match atoms left and right}$$

$$d_1 = -2, \; d_2 = 1$$

$$\lim_{t\to\infty} x_h(t) = 0 \implies$$

periodic solution equals
$x(t) = -2\cos t + \sin t$

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3. **(ch5)** Complete all parts below.

(a) [75%] Determine for $y^{v} - 9y''' = xe^{3x} + x^3 + 2\sin 3x$ the **corrected** trial solution for y_p according to the method of undetermined coefficients. To save time, **do not** evaluate the undetermined coefficients (that is, do undetermined coefficient steps [1] and [2], but skip steps [3] and [4])! Undocumented detail or guessing earns no credit.

(b) [25%] Using the *recipe* for higher order constant-coefficient differential equations, write out the general solution when the characteristic equation is $(r^2-1)^3(r+1)(r^2+6r+10)^2 = 0$.

(a) atom list of RHS $= 1, x, x^2, x^3, e^{3x}, xe^{3x}, \cos 3x, \sin 3x$

Char eq $r^5 - 9r^3 = 0$

roots $= 0, 0, 0, 3, -3$

$$y = (d_1 + d_2 x + d_3 x^2 + d_4 x^3)\, x^3 + [(d_5 + d_6 x)e^{3x}]\, x + d_7 \cos 3x + d_8 \sin 3x$$

$s = 3$ for multiplier x^s on coefficients $d_1 \to d_4$

$s = 1$ for multiplier x^s on coefficients d_5, d_6

$s = 0$ for d_7, d_8.

(b) $(r-1)^3(r+1)^4((r+3)^2+1)^2 = 0$

$$y = u_1 e^{x} + u_2 e^{-x} + (u_3 \cos x + u_4 \sin x)e^{-3x}$$

$u_1 = c_1 + c_2 x + c_3 x^2$

$u_2 = c_4 + c_5 x + c_6 x^2 + c_7 x^3$

$u_3 = c_8 + c_9 x$

$u_4 = c_{10} + c_{11} x$

11 constants because of 11 roots.

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4. **(ch6)** Complete all of the items below.

(a) [40%] Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 1 & -1 & 0 \\ 0 & 5 & -2 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 3 & 1 \end{bmatrix}$. To save time, **do not** find eigenvectors!

(b) [60%] Given $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{bmatrix}$, then there exists an invertible matrix P and a diagonal matrix D such that $AP = PD$. Find a column of P so that 4 appears in the same column of D.

(a)
$$\begin{vmatrix} 4-\lambda & 1 & -1 & 0 \\ 0 & 5-\lambda & -2 & 1 \\ 0 & 0 & 1-\lambda & -3 \\ 0 & 0 & 3 & 1-\lambda \end{vmatrix} = 0$$

$(4-\lambda)(5-\lambda)((1-\lambda)^2+9) = 0$ by cofactor expansion

$\boxed{\lambda = 4, 5, 1+3i, 1-3i}$

(b) Solve $A\vec{x} = 4\vec{x}$ for $\vec{x} \neq \vec{0}$:

$$\left(\begin{array}{ccc|c} -3 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$ mult combo

$$\left(\begin{array}{ccc|c} 3 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$ rref found

$$\begin{cases} x_1 + \frac{2}{3}x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

gen sol
$$\begin{cases} x_1 = -2t_1/3 \\ x_2 = -t_1 \\ x_3 = t_1 \end{cases}$$

$$\boxed{\text{Col} = \begin{pmatrix} -2/3 \\ -1 \\ 1 \end{pmatrix}}$$

or any scalar multiple of this answer

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5. **(ch6)** Complete all parts below.

Consider a given 3×3 matrix A having three eigenpairs

$$5, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}; \quad -3, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}; \quad 3, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$

(a) [50%] Display the vector general solution $\mathbf{x}(t)$ of the linear differential system $\mathbf{x}' = A\mathbf{x}$.

(b) [20%] Write a matrix algebra formula for the matrix A of (a) above. To save time, do not evaluate anything.

(c) [30%] Let B be a certain 2×2 matrix. Fourier's model for the computation of $B\mathbf{x}$ is known to be

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \text{implies}$$

typo Corrected in class (B)

$$A\mathbf{x} = -c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 3c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Find B.

(a) $\vec{x}(t) = c_1 e^{5t}\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + c_3 e^{3t}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

(b) $AP = PD$ $\quad P = \begin{pmatrix} 2 & -1 & 2 \\ 3 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ $\quad D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$A = PDP^{-1}$

(c) $P = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$ $\quad D = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$

$BP = PD \implies B = PDP^{-1}$

$= \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix} \cdot \frac{1}{4}$

$= \begin{pmatrix} -1 & 3 \\ -2 & -6 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix} \frac{1}{4}$

$= \begin{pmatrix} -8 & 2 \\ 8 & -8 \end{pmatrix} \frac{1}{4}$

$= \boxed{\begin{pmatrix} -2 & 1/2 \\ 2 & -2 \end{pmatrix}}$

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