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APPLIED ANALYSIS.

Memoir on the advantage of the banker at the game of trente et quarante;

By the gentleman Mr. Poisson, of the Royal Academy of Sciences, Adviser to the Royal Council of Public Instruction.

(Read at the Academy of Sciences 13 March 1820.)

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The evaluation of the chances, in games of chance, was the origin of the calculus of probabilities, and the subject of the problems first solved by Pascal, Fermat, Huygens and the other geometers who first occupied themselves with this calculus. Those who followed them in this course, and particularly Mr. Laplace, did not limit themselves to solving problems of this nature; they have extended the applications of this calculus to questions of greater interest; and the theory of probabilities has become one of the most important branches of the mathematical sciences. But, among the numerous questions, of different types, which depend on this theory, there exists one which has not yet attracted the attention of the geometers, or, at least, I have seen nowhere that they are concerned with the calculation of the chances, in the game known indifferently under the name of *trente et quarante* or under that of *trente et un*. In fact, one finds in the Encyclopedia, in the table of contents, following the description of this game, a numerical evaluation of the chances that it presents, but, with a little reflection, one easily assures oneself of the inexactitude of the principle upon which this calculation is based. However, *trente et un* being the game on which annually the most money is spent, in the public games, it would be useful to know *a priori* the advantage of the people to whom the city of Paris grants the exclusive right of this game. This question moreover, by the complication of the conditions of the game, is one of the most curious problems of probability that one could propose. Most of these problems are solved, as we know, by standard methods, based on the integration of finite and partial difference equations; but, in the question of concern here, it does not take long to recognize that the use of these equations can be of no help, and one is obliged, for the solution, to resort to new methods. Those that I have employed, in this memoir, have

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led me to some formulas whose development, in powers of one or of several variables, will make known all the chances of *trente et quarante* that one would want to determine; in the same manner that, in less complicated questions, the development of the power of a binomial, or of a polynomial composed of more than two terms, serves to find the probability of compound events, after that of simple events. The principal consequence that I have deduced is that at the game of *trente et quarante*, the advantage of the banker is very nearly equal to *eleven thousandths* of the sum of the stakes, or, in other words, that, in a very long sequence of coups, what one calls a *refait of thirty-one*, and for which the banker takes half of the sum of the stakes, should occur, very nearly, *twenty-two times* in *a thousand* coups; the probability of this proportion being able to approach certainty as close as one would want, by prolonging the game suitably.

1. The game of *trente et quarante* is played with six complete decks of cards, forming 312 cards in all. The face cards count for ten, the ace for one, and each of the other cards for the number of its points. All the cards being shuffled, one draws successively one, two, three, ... cards, until the sum of the points of the cards drawn has passed thirty; and one stops drawing as soon as it has passed this limit. One then makes a second drawing, similar to the first, that is to say, one draws in the same way, from the remaining cards, one, two, three, ... cards, until the sum of their points has surpassed thirty. These two drawings together form what is called a *coup*. After the first coup, one plays a second, in the same manner, with the remaining cards; after that a third; and so on, until one has exhausted the totality of the cards that are available. When these 312 cards are drawn, or when there are not enough remaining to make a complete coup, one has made what is called a *deal*; and the game can then recommence, following the same rules, with the same or other cards.

When one stops, during each drawing, once one has passed thirty, and since 10 is the highest number of points that one card can introduce, it follows that each drawing offers no fewer than 10 different points, of which the smallest will be 31 and the largest 40. In adding the points of all the cards which comprise six complete decks, the sum is equal to 2040. Each coup employs at most 80 and at least 62 of these points; the number of coups which will comprise a complete deal will therefore always be between 2040/80 and 2040/62, or between 25 and 32.

Before each coup is begun, each player bets against the *banker*, for one or the other of the two drawings of which this coup will be comprised. The drawing which introduces the fewest points, or is the closest to thirty, is that which wins. The banker pays to the players who have bet for that drawing a sum equal to that which they played, and he takes the stakes of the players who bet for the other drawing (*). If the two drawings lead to equal points, and greater than 31, as 32 and 32, 33 and 33, ..., 40 and 40, the coup is null:

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* Usually, to make it easier, the players, who are also called the *punters*, place their stakes on the edge of two circular boxes, one black, corresponding to the first drawing of each coup, and the other red, corresponding to the second; and from this comes that, in some localities, *trente et quarante* is also called *rouge et noir*, and that one says that one bets for the color black or for red, according to whether one bets for the first or for the second drawing.

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the banker neither pays nor receives anything for these sorts of coups; so that if this were the same for the coup 31 and 31, the banker would evidently have no advantage, and the game would be perfectly equal between the players and him. But, when there occurs what one calls a *refait* of 31, or otherwise says, if the two drawings of the same coup introduce this number, the coup is not deemed null, and the banker takes half of the stakes of all the players. The advantage of the banker, at the game of *trente et quarante*, is therefore equal, for each coup, to half the sum of all the stakes, multiplied by the probability of the refait of 31, relative to this coup. Therefore, the principal object of this memoir consists of determining this probability. But we proceed, beforehand, to solve several problems of chance which have not been treated up to now, and for which this determination will be no more than a particular application.

2. An urn contains x_1 balls carrying the number 1, x_2 balls carrying the number 2, x_3 balls carrying the number 3, ..., and finally x_i balls carrying the number i, the highest of all those with which the balls are marked; one draws successively one, two, three, ... balls, without replacement in the urn, until they are all removed; this sequence of draws continues, until the sum of the numbers introduced by the balls has attained or surpassed a given number x: one asks the probability that this sum will be equal to the number x?

Let s be the total number of balls contained in the urn; so that one would have

$$x_1 + x_2 + x_3 + \cdots + x_i = s.$$

Designate by $a_1, a_2, a_3, \dots, a_i$, numbers, whole or zero, respectively less than $x_1, x_2, x_3, \dots, x_i$, or which are all at most equal to them; and make also

$$a_1 + a_2 + a_3 + \cdots + a_i = n.$$

By the first rules of the calculus of probabilities, if one makes a number n of successive draws, without replacing the balls in the urn, the probability of introducing at first a_1 balls numbered 1, next a_2 balls numbered 2, then a_3 balls numbered 3, ..., finally a_i balls numbered i, will be expressed by[1]

$$\frac{(s-n)!}{s!} \cdot \frac{x_1!}{(x_1-a_1)!} \cdot \frac{x_2!}{(x_2-a_2)!} \cdot \frac{x_3!}{(x_3-a_3)!} \cdots \frac{x_i!}{(x_i-a_i)!}.$$

in adopting, in general, with some geometers, for typographical convenience, the expression $k!$, as the symbol of $1 \cdot 2 \cdot 3 \cdots k$. In order to deduce the probability of introducing, in any order, in n draws, a_1 balls numbered 1, a_2 balls numbered 2, a_3 balls numbered 3, ..., a_i

[1] Translator's note: Poisson had

$$\frac{(s-n)!}{s!} \cdot \frac{x_1!}{(x_1-a_1-1)!} \cdot \frac{x_2!}{(x_2-a_2-1)!} \cdot \frac{x_3!}{(x_3-a_3-1)!} \cdots \frac{x_i!}{(x_i-a_i-1)!};$$

balls numbered i, it would be necessary to multiply this quantity by the number $1 \cdot 2 \cdot 3 \cdots n$ of permutations which are compatible with the n numbers introduced, if all these numbers were different; but, because equal numbers must not be permuted among them, it will be necessary to multiply the probability previously obtained by merely

$$\frac{n!}{a_1! a_2! a_3! \cdots a_i!};$$

which will give a product which can be written thus[1]

$$\frac{n!(s-n)!}{s!} \cdot \frac{x_1!}{a_1!(x_1-a_1)!} \cdot \frac{x_2!}{a_2!(x_2-a_2)!} \cdots \frac{x_i!}{a_i!(x_i-a_i)!}.$$

But, on integrating, from $y = 0$ to $y = 1$, one has

$$\frac{n!(s-n)!}{s!} = (s+1) \int (1-y)^{s-n} y^n \, dy;$$

in doing so, to abbreviate

$$\frac{x_1! y^{a_1}}{a_1!(x_1-a_1)!(1-y)^{a_1}} \cdot \frac{x_2! y^{a_2}}{a_2!(x_2-a_2)!(1-y)^{a_2}} \cdots \frac{x_i! y^{a_i}}{a_i!(x_i-a_i)!(1-y)^{a_i}} = Y,$$

and observing that the sum of the exponents $a_1, a_2, a_3, \ldots, a_i$ is equal to n, this product will therefore become

$$(s+1) \int (1-y)^s Y \, dy.$$

The sum of the numbers introduced by this sequence of draws will be $a_1 + 2a_2 + 3a_3 + \cdots + ia_i$; but, according to the statement of the problem, this sum should be equal to x; the probability required by this statement will therefore be equal to the sum of the values that one will deduce from the expression $(s+1) \int (1-y)^s Y \, dy$ in giving to the numbers $a_1, a_2, a_3, \ldots, a_i$, all the values, including zero, which satisfy the equation

$$a_1 + 2a_2 + 3a_3 + \cdots + ia_i = x.$$

Therefore, if we designate by Y_1 the sum of all the values of Y which correspond to these values of $a_1, a_2, a_3, \ldots, a_i$, and by X the probability desired, we will have

$$X = (s+1) \int (1-y)^s Y_1 \, dy.$$

[1] Translator's note: Poisson had

$$\frac{n!(s-n)!}{s!} \cdot \frac{x_1!}{a_1!\,(x_1-a_1-1)!} \cdot \frac{x_2!}{a_2!(x_2-a_2-1)!} \cdots \frac{x_i!}{a_i!(x_i-a_i-1)!}.$$

Now, take an undetermined t, and suppose that one makes the product of the following series:

$$1+\frac{x_1}{1}\cdot\frac{yt}{1-y}+\frac{x_1}{1}\cdot\frac{x_1-1}{2}\cdot\frac{y^2t^2}{(1-y)^2}+\frac{x_1}{1}\cdot\frac{x_1-1}{2}\cdot\frac{x_1-2}{3}\cdot\frac{y^3t^3}{(1-y)^3}+\cdots,$$
$$1+\frac{x_2}{1}\cdot\frac{yt^2}{1-y}+\frac{x_2}{1}\cdot\frac{x_2-1}{2}\cdot\frac{y^2t^4}{(1-y)^2}+\frac{x_2}{1}\cdot\frac{x_2-1}{2}\cdot\frac{x_2-2}{3}\cdot\frac{y^3t^6}{(1-y)^3}+\cdots,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$1+\frac{x_i}{1}\cdot\frac{yt^i}{1-y}+\frac{x_i}{1}\cdot\frac{x_i-1}{2}\cdot\frac{y^2t^{2i}}{(1-y)^2}+\frac{x_i}{1}\cdot\frac{x_i-1}{2}\cdot\frac{x_i-2}{3}\cdot\frac{y^3t^{3i}}{(1-y)^3}+\cdots;$$

if one orders this product according to the power of t, it is evident, according to the form of Y, that the sum Y_1 of values of this quantity will be none other than the coefficient of t^x in the series that one will obtain; moreover, the i factors of the product are the developments of the powers

$$\left(1+\frac{yt}{1-y}\right)^{x_1},\ \left(1+\frac{yt^2}{1-y}\right)^{x_2},\ \ldots,\ \left(1+\frac{yt^i}{1-y}\right)^{x_i};$$

therefore, because $x_1+x_2+x_3+\cdots+x_i=s$, Y_1 will be the coefficient of t^x, in the development of the product[1]

$$(1-y)^{-s}(1-y+yt)^{x_1}(1-y+yt^2)^{x_2}\cdots(1-y+yt^i)^{x_i},$$

and consequently the probability X will also be the coefficient of t^x in the development of

$$(s+1)\int(1-y+yt)^{x_1}(1-y+yt^2)^{x_2}(1-y+yt^3)^{x_3}\cdots(1-y+yt^i)^{x_i}\,dy;$$

the integral being taken, as before, from $y=0$ to $y=1$.

3. To solve the same problem by the method of finite difference equations, I observe that the desired probability is a function of the number x and of the numbers $x_1, x_2, x_3, \ldots, x_i$, and I designate it by $Z_{x,x_1,x_2,\ldots,x_i}$. After the first draw, this function will become one of the quantities

$$\begin{gathered}Z_{x-1,x_1-1,x_2,x_3,\ldots,x_i},\\ Z_{x-2,x_1,x_2-1,x_3,\ldots,x_i},\\ Z_{x-3,x_1,x_2,x_3-1,\ldots,x_i},\\ \cdots\cdots\cdots\cdots,\\ Z_{x-i,x_1,x_2,x_3,\ldots,x_i-1},\end{gathered}$$

because the ball drawn will carry one of the numbers $1, 2, 3, \ldots, i$, respectively. The probabilities of these events are moreover respectively

$$\frac{x_1}{s},\ \frac{x_2}{s},\ \frac{x_3}{s},\ \ldots,\ \frac{x_i}{s};$$

[1] Translator's note: Poisson had $(1-y)^s$ in place of $(1-y)^{-s}$.

but, it is evident that the probability before the first draw is equal to the sum of the probabilities which will take place afterwards, each multiplied by the probability of the event to which it corresponds: it is this principle which serves to put in the form of an equation most of the problems of probability; and, in the present question, we have as a result

$$\begin{aligned} Z_{x,x_1,x_2,x_3,\dots,x_i} = {} & \frac{x_1}{s} Z_{x-1,x_1-1,x_2,x_3,\dots,x_i} \\ & + \frac{x_2}{s} Z_{x-2,x_1,x_2-1,x_3,\dots,x_i} \\ & + \frac{x_3}{s} Z_{x-3,x_1,x_2,x_3-1,\dots,x_i} \\ & + \cdots\cdots\cdots\cdots \\ & + \frac{x_i}{s} Z_{x-i,x_1,x_2,x_3,\dots,x_i-1}. \end{aligned} \tag{1}$$

The same reasoning shows that, if one has $x = a$, a being smaller than i, one will have

$$\begin{aligned} Z_{a,x_1,x_2,x_3,\dots,x_i} = {} & \frac{x_1}{s} Z_{a-1,x_1-1,x_2,x_3,\dots,x_i} \\ & + \frac{x_2}{s} Z_{a-2,x_1,x_2-1,x_3,\dots,x_i} \\ & + \frac{x_3}{s} Z_{a-3,x_1,x_2,x_3-1,\dots,x_i} \\ & + \cdots\cdots\cdots\cdots \\ & + \frac{x_{a-1}}{s} Z_{1,x_1,x_2,x_3,\dots,x_{a-1}-1,\dots,x_i} \\ & + \frac{x_a}{s}. \end{aligned}$$

In comparing this equation to the preceding, one sees that, for it to be included, and in order that the equation (1) still holds, for all the values of $x < i$, it is necessary to suppose the function Z equals unity, when $x = 0$, and zero, for all the negative values of x smaller than i, leaving the sign out of account, whatever furthermore may be the other variables $x_1, x_2, x_3, \dots, x_i$. That being so, in making successively $x = 1$, $x = 2$, $x = 3$, ..., in equation (1), one will deduce from it[1]

$$\begin{aligned} Z_{1,x_1,x_2,\dots,x_i} &= \frac{x_1}{s}, \\ Z_{2,x_1,x_2,\dots,x_i} &= \frac{x_1}{s} Z_{1,x_1-1,x_2,\dots,x_i} + \frac{x_2}{s}, \\ Z_{3,x_1,x_2,\dots,x_i} &= \frac{x_1}{s} Z_{2,x_1-1,x_2,\dots,x_i} + \frac{x_2}{s} Z_{1,x_1,x_2-1,\dots,x_i} + \frac{x_3}{s}, \\ & \cdots\cdots\cdots\cdots\cdots\cdots, \end{aligned}$$

[1] Translator's note: Poisson had $Z_{3,x_1,x_2-1,\dots,x_i}$ in place of $Z_{1,x_1,x_2-1,\dots,x_i}$ in the third equation.

putting also successively $x_1 - 1$, $x_2 - 1$, $x_3 - 1$, ..., in the place of x_1, x_2, x_3, ..., in the first of these equations, one will be able then to eliminate the quantities

$$Z_{1,x_1-1,x_2,x_3,\ldots,x_i},$$
$$Z_{1,x_1,x_2-1,x_3,\ldots,x_i},$$
$$Z_{1,x_1,x_2,x_3-1,\ldots,x_i},$$
$$\ldots\ldots\ldots\ldots,$$

contained in the others, and one will have

$$Z_{2,x_1,x_2,\ldots,x_i} = \frac{x_1}{s}\cdot\frac{x_1-1}{s-1} + \frac{x_2}{s},$$
$$Z_{3,x_1,x_2,\ldots,x_i} = \frac{x_1}{s}Z_{2,x_1-1,x_2,\ldots,x_i} + \frac{x_2}{s}\cdot\frac{x_1}{s-1} + \frac{x_3}{s},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;$$

and likewise, by the method of the latter, one will obtain

$$Z_{3,x_1,x_2,\ldots,x_i} = \frac{x_1}{s}\cdot\frac{x_1-1}{s-1}\cdot\frac{x_1-2}{s-2} + 2\frac{x_2}{s}\cdot\frac{x_1}{s-1} + \frac{x_3}{s},$$

and so on.

In this manner, one will easily calculate the desired probability, when x is a small number; but the calculation becomes impractical, when this number is a little more considerable; and it will be necessary then to resort to the general integral of equation (1). This linear equation has variable coefficients; nevertheless, if one multiplies all of its terms by s, its coefficients will include the variables only to the first degree; in this case it will be possible to integrate equation (1) by the method of definite integrals. But this method will only very difficultly lead to the solution of the problem that we have proposed; that is why we will limit ourselves to verifying that the solution that we have found satisfies equation (1).

4. Let, in order to arrive there,

$$\sum t^x Z_{x,x_1,x_2,x_3,\ldots,x_i} = T_{x_1,x_2,x_3,\ldots,x_i};$$

$\sum$ indicating a sum which extends to all positive whole values of x, including $x = 0$, and up to $x = \infty$. The values of the function Z which correspond to negative x being zero, according to what was said before, one will have

$$\sum t^x Z_{x-i',x_1,x_2,\ldots,x_i} = t^{i'} T_{x_1,x_2,\ldots,x_i},$$

i' being a positive whole number; and, because this function[1] Z is equal to one when $x = 0$, the first term of the function T will also be equal to one. If therefore we multiply equation

[1] Translator's note: Poisson had i' in place of Z.

(1) by[1] t^x, that we might give to x all the values from $x = 1$ up to $x = \infty$, and that we might make the sum of all the equations which correspond to these values, we will have for a result[2]

$$\begin{aligned} T_{x_1,x_2,\ldots,x_i} - 1 = \frac{tx_1}{s} & T_{x_1-1,x_2,x_3,\ldots,x_i} \\ & + \frac{t^2x_2}{s} T_{x_1,x_2-1,x_3,\ldots,x_i} \\ & + \frac{t^3x_3}{s} T_{x_1,x_2,x_3-1,\ldots,x_i} \\ & + \cdots\cdots\cdots\cdots \\ & + \frac{t^ix_i}{s} T_{x_1,x_2,x_3,\ldots,x_i-1}. \end{aligned} \tag{2}$$

Moreover, in regard to the expression of X, found at the end of no. 1, and which should also be the value of the function Z, one sees that one should have

$$T_{x_1,x_2,x_3,\ldots,x_i} = (s+1) \int (1-y+yt)^{x_1}(1-y+yt^2)^{x_2} \cdots (1-y+yt^i)^{x_i}\, dy,$$

the integral being taken from $y = 0$ to $y = 1$; the question consists therefore of verifying that this value of the function T satisfies equation (2), regardless of the value of t.

But, I make

$$\frac{y}{1-y} = u, \quad \text{or} \quad y = \frac{u}{1+u}, \quad \text{and} \quad dy = \frac{du}{(1+u)^2};$$

and, to abbreviate,

$$(1+ut)^{x_1} \cdot (1+ut^2)^{x_2} \cdot (1+ut^3)^{x_3} \cdots (1+ut^i)^{x_i} = U;$$

the preceding equation becomes

$$T_{x_1,x_2,x_3,\ldots,x_i} = (s+1) \int \frac{U\, du}{(1+u)^{s+2}};$$

and the integral should be taken from $u = 0$ to $u = \infty$. In putting successively, in this preceding equation $x_1 - 1$, $x_2 - 1$, $x_3 - 1$, …, $x_i - 1$ in place of x_1, x_2, x_3, ..., x_i, and, at the same time, $s - 1$ in place of s, one will have

$$\begin{aligned} T_{x_1-1,x_2,\ldots,x_i} &= s \int (1-y+yt)^{x_1-1} \cdot (1-y+yt^2)^{x_2} \cdots (1-y+yt^i)^{x_i}\, dy, \\ T_{x_1,x_2-1,\ldots,x_i} &= s \int (1-y+yt)^{x_1} \cdot (1-y+yt^2)^{x_2-1} \cdots (1-y+yt^i)^{x_i}\, dy, \\ & \cdots\cdots\cdots\cdots\cdots\cdots, \\ T_{x_1,x_2,\ldots,x_i-1} &= s \int (1-y+yt)^{x_1} \cdot (1-y+yt^2)^{x_2} \cdots (1-y+yt^i)^{x_i-1}\, dy; \end{aligned}$$

[1] Translator's note: Poisson had tx in place of t^x.

[2] Translator's note: Poisson neglected the factor t^ix_i/s.

but, in differentiating U with respect to u, one has

$$\begin{aligned}\frac{dU}{du} &= tx_1(1+ut)^{x_1-1}\cdot(1+ut^2)^{x_2}\cdots(1+ut^i)^{x_i}\\ &\quad+t^2x_2(1+ut)^{x_1}\cdot(1+ut^2)^{x_2-1}\cdots(1+ut^i)^{x_i}\\ &\quad+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &\quad+t^ix_i(1+ut)^{x_1}\cdot(1+ut^2)^{x_2}\cdots(1+ut^i)^{x_i-1};\end{aligned}$$

and, if one compares this equation with the preceding ones, one then easily concludes

$$\begin{aligned}s\int\frac{dU}{(1+u)^{s+1}} &= tx_1T_{x_1-1,x_2,x_3,\dots,x_i}\\ &\quad+t^2x_2T_{x_1,x_2-1,x_3,\dots,x_i}\\ &\quad+\cdots\cdots\cdots\cdots\cdots\cdots\\ &\quad+t^ix_iT_{x_1,x_2,x_3,\dots,x_i-1}.\end{aligned}$$

By means of these results, equation (2) becomes

$$(s+1)\int\frac{U\,du}{(1+u)^{s+2}}-1=\int\frac{dU}{(1+u)^{s+1}};\tag{3}$$

but, by integrating by parts, it becomes

$$\int\frac{dU}{(1+u)^{s+1}}=\frac{U}{(1+u)^{s+1}}+(s+1)\int\frac{U\,du}{(1+u)^{s+2}};$$

at the limit $u=0$, one has $U=1$; at the limit $u=\infty$, one has

$$\frac{U}{(1+u)^{s+1}}=0,$$

because U is the product of s factors of the first degree with respect to u; one will therefore have

$$\int\frac{dU}{(1+u)^{s+1}}=-1+(s+1)\int\frac{U\,du}{(1+u)^{s+2}};$$

this is identical to equation (3) which it was our concern to verify.

5. Now, let us propose this second problem: the same things being granted as in the first, one makes a first sequence of draws that one continues until the sum of the numbers introduced has attained or surpassed the number x; then, without replacing the numbers taken out, one makes a second sequence of draws, that one prolongs until the sum of the numbers introduced has likewise attained or surpassed a number given as x', one asks for the probability that one will obtain, at the same time, the sum x, in the first operation, and the sum x', in the second?

As in the first problem, the probability of introducing, in the first sequence of draws and in any order, a_1 balls numbered 1, a_2 balls numbered 2, a_3 balls numbered 3, …, a_i balls numbered i, will be expressed by

$$\frac{n!(s-n)!}{s!}\cdot\frac{x_1!}{a_1!(x_1-a_1)!}\cdot\frac{x_2!}{a_2!(x_2-a_2)!}\cdot\frac{x_3!}{a_3!(x_3-a_3)!}\cdots\frac{x_i!}{a_i!(x_i-a_i)!};$$

this event having taken place, the probability of introducing in the second sequence of draws, also in any order, b_1 balls numbered 1, b_2 balls numbered 2, b_3 balls numbered 3, …, b_i balls numbered i, will be

$$\frac{n'!(s-n-n')!}{(s-n)!}\cdot\frac{(x_1-a_1)!}{b_1!(x_1-a_1-b_1)!}\cdot\frac{(x_2-a_2)!}{b_2!(x_2-a_2-b_2)!}\cdots\frac{(x_i-a_i)!}{b_i!(x_i-a_i-b_i)!};$$

in respectively designating by n and n', the number of numbers which comprise the first and second drawings, that is to say, in making

$$a_1+a_2+a_3+\cdots+a_i=n,$$
$$b_1+b_2+b_3+\cdots+b_i=n'.$$

The probability of the succession of these two events one after the other will be the product of the two probabilities which correspond to them, which product can be written thus

$$\frac{(n+n')!(s-n-n')!}{s!}\cdot\frac{n!n'!}{(n+n')!}$$
$$\cdot\frac{x_1!}{a_1!b_1!(x_1-a_1-b_1)!}\cdot\frac{x_2!}{a_2!b_2!(x_2-a_2-b_2)!}\cdots\frac{x_i!}{a_i!b_i!(x_i-a_i-b_i)!};$$

but, in integrating from $y=0$ to $y=1$, and from $z=0$ to $z=1$, we have

$$\frac{(n+n')!(s-n-n')!}{s!}=(s+1)\int(1-y)^{s-n-n'}y^{n+n'}\,dy,$$

$$\frac{n!n'!}{(n+n')!}=(n+n'+1)\int(1-z)^{n'}z^n\,dz;$$

furthermore in making $\alpha=1$, after differentiation, we also have

$$n+n'+1=\frac{d\alpha^{n+n'+1}}{d\alpha};$$

that which changes the last equation into this

$$\frac{n!n'!}{(n+n')!}=\frac{d}{d\alpha}\int(1-z)^{n'}z^n\alpha^{n+n'+1}\,dz,$$

and, by means of these values and of those of n and n', the preceding product will be equal to

$$(s+1)\frac{d}{d\alpha}\int\int P(1-y)^s\alpha\,dy\,dz,$$

in making, to abbreviate,

$$X_1\cdot X_2\cdot X_3\cdots X_i=P$$

where one designates by X_m, in general, the product[1]

$$\frac{x_m!y^{a_m+b_m}(1-z)^{b_m}z^{a_m}\alpha^{a_m+b_m}}{a_m!b_m!(x_m-a_m-b_m)!(1-y)^{a_m+b_m}};$$

the index m being one of the numbers between 1 and i inclusive.

The statement of the problem requires that one have

$$\begin{aligned}a_1+2a_2+3a_3+\cdots+ia_i&=x,\\ b_1+2b_2+3b_3+\cdots+ib_i&=x';\end{aligned}$$

therefore, if one gives to the nonnegative integers a_1, a_2, a_3, ..., a_i, b_1, b_2, b_3, ..., b_i all the possible values which satisfy these conditions, and one calls P_1 the sum of the values corresponding to P, and p the probability required by this same statement, one will have

$$p=(s+1)\frac{d}{d\alpha}\int\int(1-y)^sP_1\alpha\,dy\,dz.$$

But, t and θ being two different unknowns, one can regard X_m as the coefficient of the product $t^{ma_m}\theta^{mb_m}$, in the development of the power

$$\left\{1+\frac{y\alpha}{1-y}[zt^m+(1-z)\theta^m]\right\}^{x_m};$$

from which one concludes that P_1 will be the coefficient of $t^x\theta^{x'}$, in the development of the product

$$\left\{1+\frac{y\alpha}{1-y}[zt+(1-z)\theta]\right\}^{x_1}\cdot\left\{1+\frac{y\alpha}{1-y}[zt^2+(1-z)\theta^2]\right\}^{x_2}\cdots\left\{1+\frac{y\alpha}{1-y}[zt^i+(1-z)\theta^i]\right\}^{x_i},$$

ordered following the powers and products of powers of t and θ; therefore finally, because $s=x_1+x_2+x_3+\cdots+x_i$, the probability p, that it is a question of determining will be the coefficient of $t^x\theta^{x'}$, in the development of this quantity

$$\begin{aligned}(s+1)\frac{d}{d\alpha}\int\int\{1-y+y\alpha[zt+(1-z)\theta]\}^{x_1}\cdot\{1-y+y\alpha[zt^2+(1-z)\theta^2]\}^{x_2}\\ \cdots\{1-y+y\alpha[zt^i+(1-z)\theta^i]\}^{x_i}\alpha\,dy\,dz;\end{aligned}$$

[1] Translator's note: Poisson has $b!$ in place of $b_m!$ in the denominator.

while remembering that one should make $\alpha = 1$, after the indicated differentiation, and take the integrals from $y = 0$ and $z = 0$, to $y = 1$ and $z = 1$.

If, instead of one or of two sequences of draws, as in the first or the second problem that we have just solved, there were three or a larger number, the preceding analysis would lead to the solution of the question; but it suffices for the principal object of this memoir to have considered the case of two sequences of draws, which is indeed that which is presented in *trente et quarante*, according to the rules of the game, stated above.

6. In order to apply the solution of the second problem to the calculation of the various chances of *trente et quarante*, it suffices to replace the numbered balls that we have considered by cards that carry the numbers marked by their number of points, which extend, consequently, from 1 to 10; the face cards counting for this last number. One will have, for example, the probability of the *refait* of 31, at any coup of the deal, in making $x = x' = 31$, and taking for $x_1, x_2, x_3, \ldots, x_i$, the numbers of cards of each type which have not come out in preceding coups; so that this probability, and the advantage of the banker which depends on it, will vary from one coup to another in the same deal, because of the cards already removed. But it is important to observe that, in order to determine the advantage of the banker, before the game begins, that is to say, the fraction of each stake that one must abandon to him so that he renounces this advantage throughout the duration of the game, it suffices to calculate the probability of the *refait* of 31, at the first coup only, or when the six decks of cards with which one is playing are still complete. Indeed, when these cards have been shuffled, if there exists any chance that an event A occurs at the first coup and an event B at another coup, at the tenth, for example; there is exactly the same chance that the event B occurs at the first coup and the event A at the tenth; because one can form another arrangement of all the cards, which only differs from that which chance has given in that the cards which are removed at the first coup are replaced by those taken out at the tenth, and *vice versa*; and, as the two arrangements are equally likely, the result is that before the game begins the probability of a *refait* of 31 is the same for the first coup, for the tenth, or for any other coup. It only varies, throughout the duration of the game, for the players who have knowledge of the cards removed; but a player who does not know the cards, would bet the same sum at all the coups, for the arrival of a *refait* of 31.

This very simple consideration makes possible the determination of the advantage of the banker, in the reduction to the calculation of a single chance which takes place at the first coup. Observe also that one could determine this advantage, in calculating the probability of the *refait* of 31, at another coup chosen at will; but it would be necessary to make a hypothesis as to the cards removed at the preceding coups, and to multiply the probability that one would find by that of this hypothesis, which would make the calculation extremely complicated.

7. At the first coup, that we have confined ourselves to considering, there are 312 cards, the first nine numbers $x_1, x_2, x_3, \ldots, x_9$, are all equal to 24; the tenth alone x_{10} is different, and quadruple of each of the first. Thus, it will be necessary to make

$$s = 312, \qquad x_1 = x_2 = x_3 = \cdots = x_8 = x_9 = 24, \qquad x_{10} = 96.$$

Furthermore, to abbreviate, let

$$\begin{aligned} 1-\alpha[zt+(1-z)\theta] &= z_1, \\ 1-\alpha[zt^2+(1-z)\theta^2] &= z_2, \\ 1-\alpha[zt^3+(1-z)\theta^3] &= z_3, \\ \cdots\cdots\cdots\cdots\cdots\cdots &, \\ 1-\alpha[zt^{10}+(1-z)\theta^{10}] &= z_{10}; \end{aligned}$$

then

$$(1-yz_1)^{x_1}(1-yz_2)^{x_2}(1-yz_3)^{x_3}\cdots(1-yz_{10})^{x_{10}} = Y;$$

and finally

$$U = (s+1)\frac{d}{d\alpha}\int\int Y\alpha\,dy\,dz,$$

where one will make $\alpha = 1$, after the differentiation, in integrating then from $y = 0$ and $z = 0$, to $y = 1$ and $z = 1$. The question is reduced to calculating the coefficient of $t^{31}\theta^{31}$, in the development of this quantity U, following the powers and products of powers of t and θ. The number of the factors of Y, and the greatness of their exponents, will make impractical the rigorous calculation of this coefficient; but one can reduce the integral $\int Y\,dy$ in a convergent series, by means of which one will obtain, to such degree of approximation that one would want, the value of the desired coefficient.

For that, let

$$V = -\frac{Y\,dy}{dY};$$

we will have

$$\int Y\,dy = -\int V\,dY;$$

and, with a sequence of integrations by parts, we will conclude

$$\begin{aligned} \int Y\,dy = Y_0\left[V_0 + V_0\frac{dV_0}{dy} + V_0\frac{d\left(V_0\frac{dV_0}{dy}\right)}{dy} + V_0\frac{d\left(V_0\frac{d\left(V_0\frac{dV_0}{dy}\right)}{dy}\right)}{dy} + \cdots\right], \\ - Y_1\left[V_1 + V_1\frac{dV_1}{dy} + V_1\frac{d\left(V_1\frac{dV_1}{dy}\right)}{dy} + V_1\frac{d\left(V_1\frac{d\left(V_1\frac{dV_1}{dy}\right)}{dy}\right)}{dy} + \cdots\right]; \end{aligned}$$

the indices 0 and 1 indicating respectively that it is necessary to make $y = 0$ and $y = 1$, after differentiations. This series is one of those that Mr. Laplace has given, to calculate by approximation, the integrals of the functions of large numbers. It is easy to assure oneself that the second part, which corresponds to $y = 1$, would give in the following development the powers and products of powers of t and θ, only the terms in which the sum of the exponents of these variables would surpass $x_1 + 2x_2 + 3x_3 + \cdots + 10x_{10}$; so that we ought to make abstraction of it in the present question; therefore, because $Y_0 = 1$, we will simply have

$$\int Y\,dy = V_0\left\{1 + \frac{dV_0}{dy} + \frac{d\left(V_0\frac{dV_0}{dy}\right)}{dy} + \frac{d\left(V_0\frac{d\left(V_0\frac{dV_0}{dy}\right)}{dy}\right)}{dy} + \cdots\right\}.$$

We will also have

$$V = 1\Bigg/\left(\frac{x_1z_1}{1-yz_1} + \frac{x_2z_2}{1-yz_2} + \frac{x_3z_3}{1-yz_3} + \cdots + \frac{x_{10}z_{10}}{1-yz_{10}}\right);$$

If therefore we develop the denominator of this expression following the powers of y, and that, m being an arbitrary number, we would have[1]

$$x_1z_1^m + x_2z_2^m + x_3z_3^m + \cdots + x_{10}z_{10}^m = Z_m,$$

it will result that

$$V = \frac{1}{Z_1 + Z_2y + Z_3y^2 + Z_4y^3 + \cdots};$$

and, in substituting this value in that of $\int Y\,dy$, one will find

$$\int Y\,dy = \frac{1}{Z_1} - \frac{Z_2}{Z_1^3} + \frac{3Z_2^2 - 2Z_1Z_3}{Z_1^5} - \frac{15Z_2^3 - 20Z_1Z_2Z_3 + 6Z_1^2Z_4}{Z_1^7} + \cdots.$$

Because of the number and of the greatness of the quantities $x_1, x_2, x_3, \ldots, x_{10}$, that they contain, the expressions $Z_1, Z_2, Z_3, \ldots$ have large values, and all of the same order of magnitude, from which it results that this series is highly convergent, at least in its first terms; because the numerical coefficients which are found in their numerators cross indefinitely, and finish by making this series divergent. But, in keeping to the part in which it is convergent, one will have an expression very close to $\int Y\,dy$; and, if we at first confine ourselves to its first term, we will have[2]

$$U = (s+1)\frac{d}{d\alpha}\int\frac{\alpha\,dz}{Z_1},$$

for the value corresponding to U.

The value of Z_1 is the same thing as[3]

$$Z_1 = s - \alpha zT - \alpha(1-z)\Theta,$$

in making, for abbreviation,

$$T = x_1t + x_2t^2 + x_3t^3 + \cdots + x_{10}t^{10},$$
$$\Theta = x_1\theta + x_2\theta^2 + x_3\theta^3 + \cdots + x_{10}\theta^{10}.$$

[1] Translator's note: Poisson had $x_{10}z_{10}$ in place of $x_{10}z_{10}^m$.
[2] Translator's note: Poisson had dZ in place of dz.
[3] Translator's note: Poisson had αT in place of αzT.

If one substitutes it in U, and one makes $\alpha = 1$, after having carried out the indicated differentiation, one will have

$$U = \int \frac{(1+s)s\,dz}{[s - zT - (1-z)\Theta]^2};$$

and, in integrating, from $z = 0$ to $z = 1$, it happens that

$$U = \frac{(s+1)s}{T - \Theta}\left(\frac{1}{s-T} - \frac{1}{s-\Theta}\right) = \frac{s+1}{s}\left(1 - \frac{T}{s}\right)^{-1}\left(1 - \frac{\Theta}{s}\right)^{-1}.$$

But, the two symbols T and Θ representing similar functions of t and θ, the coefficients of the same powers of their respective variables will be the same in the development of the two powers

$$\left(1 - \frac{T}{s}\right)^{-1}, \qquad \left(1 - \frac{\Theta}{s}\right)^{-1};$$

designating therefore by k the coefficient of t^{31}, in the first of the two developments, and calling p the probability of the refait of 31, or the coefficient of $t^{31}\theta^{31}$, in the development of U, we will have

$$p = \frac{s+1}{s}k^2;$$

so that it remains to calculate only the numerical value of this coefficient k.

8. Before doing this calculation, we will observe that the quantity k would be the probability of simple 31, and, consequently, k^2 that of the refait of 31, if one puts the cards back in the deck, as they are removed. Indeed, in this hypothesis, it is easy to see, according to the theory of combinations, that the probability of bringing about a given sum x, in a specified number m of successive draws, is nothing other than the coefficient of t^x; in the development of the power

$$\left(\frac{x_1}{s}t + \frac{x_2}{s}t^2 + \frac{x_3}{s}t^3 + \cdots + \frac{x_{10}}{s}t^{10}\right)^m,$$

or of T^m/s^m; consequently, the probability of bringing about this same sum in any number of draws will be the coefficient of t^x in the development of the series

$$\frac{T}{s} + \frac{T^2}{s^2} + \frac{T^3}{s^3} + \frac{T^4}{s^4} + \cdots,$$

that one can stop at will at the xth term or continue to infinity; one can also add to it unity; and then it becomes equal to $1/(1 - T/s)$, which it was necessary to demonstrate.

It results from this remark that because of the large number of cards that one employs at *trente et quarante*, the probability of 31 at the first coup is little different from what it would be, if one subjected oneself to put back the cards in the *pile*, as they are removed. This result which would also take place at the first coup, for all the other chances of the game, is evident by itself; and one can consider it as a verification of the method of approximation employed in the preceding number.

9. Let us presently make use of the numerical values of $s, x_1, x_2, x_3, \dots, x_{10}$. We will have

$$\frac{T}{s} = a(t + t^2 + t^3 + \cdots + t^9 + 4t^{10}) = a \cdot \frac{t - t^{11}}{1 - t} + 3at^{10};$$

where one has made, to abbreviate, [1]

$$\frac{24}{s} = \frac{24}{312} = \frac{1}{13} = a.$$

One concludes from there

$$\frac{1}{1 - \frac{T}{s}} = \frac{1 - t}{1 - (1 + a)t - a(3 - 4t)t^{10}};$$

and, developing it first following the powers of t^{10}, it becomes

$$\frac{1}{1 - \frac{T}{s}} = \frac{1 - t}{1 - (1 + a)t} + \frac{a(1 - t)(3 - 4t)t^{10}}{[1 - (1 + a)t]^2} + \frac{a^2(1 - t)(3 - 4t)^2 t^{20}}{[1 - (1 + a)t]^3} + \frac{a^3(1 - t)(3 - 4t)^3 t^{30}}{[1 - (1 + a)t]^4}.$$

One rejects the terms following of this series, which would contain the power t^{40} and some larger powers of t. It is easy to take the coefficient of t^{31}, in each of the four terms that one preserves; and, in making the sum of these four coefficients, one will have the exact value of k; to be precise:

$$k = a(1 + a)^{30} + \frac{a}{1} \cdot \frac{d[(1 + a)^{20}(3a - 1)a]}{da} + \frac{a^2}{1 \cdot 2} \cdot \frac{d^2[(1 + a)^{10}(3a - 1)^2 a]}{da^2} + \frac{a^3}{1 \cdot 2 \cdot 3} \cdot \frac{d^3[(3a - 1)^3 a]}{da^3}.$$

I carry out the indicated differentiations; then by means of $a = 1/13$, I reduce this formula in numbers; and, in pushing the approximation up to the decimals of the sixth order, I find

$$k = 0.148062;$$

from which it results

$$p = \frac{313}{312} k^2 = 0.021993.$$

The calculation would become very laborious, if one would want to have consideration for the second term, and for the terms following in the development of $\int Y\, dy$, that we have neglected. In taking account of the second term, I have found that the first value approaching p must be diminished by 0.000026; the terms following would not change that value in the first six decimals; therefore, one can take

$$p = 0.021967,$$

[1] Translator's note: Poisson had 213 in place of 312.

for the probability of the refait of 31, at the beginning of the game.

According to what one has said above, one will have the advantage of the banker, by multiplying this value of p by half the sum of all the stakes; so that, in order to compensate for this advantage, before the game begins, each player would have to agree to give to the banker, at every coup, including the null coups, eleven thousandths, very nearly, of the money that he would want to risk.

10. The problems of numbers 2 and 5 also include the other chances of *trente et quarante*. If one wants, for example, to calculate the probability of bringing about the number 32, one will suppose first, in the problem of number 2, that one continues the sequence of draws, until one has attained or exceeded 32; thus, one will make the limit $x = 32$, and one will determine, by the analysis of this number, the probability of bringing about this limit; but, this probability will still not be that which it is necessary to know; because, according to the conditions of the game, one must bring about the number 32, without having passed by the number 31. But, the probability of this event is evidently equal to that which one will have calculated, as one has just said, less the probability of bringing about first 31 and then 1, which probability will be calculated by the analysis of number 5, in taking $x = 31$ and $x' = 1$. These calculations, already very long in relation to the number 32, will be even more so for the larger numbers $33, 34, 35, \ldots$. But, if one wants to know the probabilities of bringing about these different numbers, only at the first coup, one will be able to suppose that one puts the cards back in the deck as they are removed (no. 8); the calculations will thus be easy to execute, and the error that one will commit will be very small, in the same way that one saw by the example of the calculation relative to 31.

Therefore we adopt this approximate hypothesis, and we designate generally by k_n the coefficient of t^{30+n}, in the development of $1/(1 - T/s)$, n being one of the numbers $1, 2, 3, \ldots, 10$. This coefficient will express the probability of bringing about $30 + n$, in continuing the draws, until one has attained or exceeded this number. Let, at the same time, p_n be the probability of bringing about this same number, without passing by one of the smaller numbers $31, 32, \ldots, 30 + n - 1$; when the quantity k_n will have been calculated for the ten values of n from $n = 1$ to $n = 10$, it will be easy to infer the ten values corresponding to p_n, which are those that it is a question of determining.

Indeed, since one returns the cards to the deck, as they are removed, the probability of obtaining an ace, after having brought about 31, will be equal to $(1/13)k_1$, by the ordinary rule of compound events; consequently, one will have $p_2 = k_2 - (1/13)k_1$. Also, the probability of obtaining an ace after having brought about 32, in any manner whatever, and that of obtaining a *two*, after having brought about 31, will be $(1/13)k_1$ and $(1/13)k_2$; but, in subtracting these two probabilities from that which is represented by k_3, one will have the probability of bringing about 33, without having passed either by 31 or by 32; we will have therefore $p_3 = k_3 - (1/13)k_2 - (1/13)k_1$. On continuing with this reasoning, one will form this sequence of equations [1]

$$p_1 = k_1,$$

[1] Translator's note: In place of the second and third equations, Poisson had $p_2 = k_2 - (1/13)(k_1 + k_2)$ and $p_3 = k_3 - (1/13)(k_1 + k_2 + k_3)$.

$$
\begin{aligned}
p_2 &= k_2 - \frac{1}{13}k_1,\\
p_3 &= k_3 - \frac{1}{13}(k_1 + k_2),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
p_{10} &= k_{10} - \frac{1}{13}(k_1 + k_2 + k_3 + \cdots + k_9),
\end{aligned}
$$

by means of which the whole calculation is reduced to determining the numerical values of the ten quantities $k_1, k_2, k_3, \ldots, k_{10}$, of which the first is already known.

11. In order to obtain them, develop, as in no. 9, the function $1/(1 - T/s)$, following the powers of t^{10}, and continue the development, up to the power t^{40} inclusive; we will have

$$
\begin{aligned}
\frac{1}{1-\frac{T}{s}} = \frac{1-t}{1-(1+a)t} &+ \frac{a(1-t)(3-4t)t^{10}}{[1-(1+a)t]^2} + \frac{a^2(1-t)(3-4t)^2t^{20}}{[1-(1+a)t]^3}\\
&+ \frac{a^3(1-t)(3-4t)^3t^{30}}{[1-(1+a)t]^4} + \frac{a^4(1-t)(3-4t)^4t^{40}}{[1-(1+a)t]^5},
\end{aligned}
$$

while making as above, $a = 24/s = 1/13$. Then develop each term, following the powers of t, and taking the sum of the coefficients of t^{30+n}, it becomes

$$
\begin{aligned}
k_n = a(1+a)^{30+n-1} &+ \frac{a}{1}\cdot\frac{d[(1+a)^{20+n-1}(3a-1)a]}{da}\\
&+ \frac{a^2}{1\cdot 2}\cdot\frac{d^2[(1+a)^{10+n-1}(3a-1)^2a]}{da^2} + \frac{a^3}{1\cdot 2\cdot 3}\cdot\frac{d^3[(1+a)^{n-1}(3a-1)^3a]}{da^3},
\end{aligned}
$$

if n is one of the numbers $1, 2, 3, \ldots, 9$; but, in the case of $n = 10$, it will be necessary to add to this value of k_n the quantity $81a^4$, equal to 0.002836, to a millionth, which comes from the fifth term of the preceding expression, which has $81a^4t^{40}$ for the first term of its development following the powers of t. By means of this formula one will find, in stopping at the decimals of the 5th order

$$
\begin{aligned}
k_1 &= 0.14806,\\
k_2 &= 0.14930,\\
k_3 &= 0.15039,\\
k_4 &= 0.15133,\\
k_5 &= 0.15213,\\
k_6 &= 0.15278,\\
k_7 &= 0.15329,\\
k_8 &= 0.15365,\\
k_9 &= 0.15386,\\
k_{10} &= 0.15392 + 0.00284 = 0.15676;
\end{aligned}
$$

from which one concludes

$$\begin{aligned} p_1 &= 0.14806, \\ p_2 &= 0.13791, \\ p_3 &= 0.12751, \\ p_4 &= 0.11689, \\ p_5 &= 0.10605, \\ p_6 &= 0.09500, \qquad\qquad \text{sum} = 0.99999(^*). \\ p_7 &= 0.08375, \\ p_8 &= 0.07232, \\ p_9 &= 0.06072, \\ p_{10} &= 0.05178, \end{aligned}$$

As one of the ten events to which these probabilities correspond must necessarily occur, the sum of their values must be equal to unity, which effectively happens, to about one hundred thousandth.

These values of $p_1, p_2, p_3, \dots, p_{10}$, will serve to regulate the fate or the *parti* of the players, after the first of the two drawings of which a coup is composed. Suppose, for example, that this drawing had led to the point 34, and let a be the stake of a player who has bet for the second drawing; if there arrives the point 34 at the second drawing, the coup is null, which is worth a for the player; if there arrives a point less than the point 34, the player will have won, and he will receive $2a$; finally, if there arrives a point greater than 34, he will have lost and will receive nothing. That which it would be necessary to give him, if the player renounces the second drawing[1], calculated according the rule of *mathematical expectation*, is therefore equal to $(p_4 + 2p_3 + 2p_2 + 2p_1)a$, one has $(0.94387)a$, thus, he will

(*) In the article from the Encyclopedia cited at the beginning of this memoir, one has also determined these same probabilities, in supposing that they are between them as the numbers of different cards which can end at the points to which they correspond. Thus, for example, 31 can be terminated by any of the cards of the deck, from *ace* to the *ten*, while the point 40 can only be finished by a *ten*; and, as at the beginning of the deck, there are 13 cards of different numbers, of which four only are *ten*, it would result, following the article cited, that the probability of 31 should be equal to three times and a quarter that of the point 40; which does not agree with the results that we find; and it would be the same for the probabilities of the other points. But it is easy to see that this reasoning is inexact. Indeed, it is very true that the 31 can come from a drawing which will have led first to 30 and then 1, or just as well 29 and then 2, or 28 and then 3, ... or finally 21 and then 10. It is equally true that the point 40 can only come from a drawing which leads first to 30 and then 10; but, in order that the probabilities of these events be between them like the respective numbers 13 and 4, it would be necessary that the probabilities of bringing about the numbers $21, 22, 23, \dots, 30$, in the first part of the drawing be equal, which does not happen, as one can assure oneself. (Note of Mr. Poisson).

[1] Translator's note: Poisson used *coup* here, mistakenly it appears.

have already lost $(0.05613)a$, or about *fifty-six thousandths* of his stake. When the first drawing has led to 35, the coup is to the advantage of the players who have bet for the second drawing, and their *parti* is equal to $(1.16681)a$; the stake always being represented by a.

The squares $p_2^2, p_3^2, p_4^2, \ldots, p_{10}^2$, will be the probabilities of the null coups 32 and 32, 33 and 33, 34 and 34, ..., 40 and 40, calculated always under the hypothesis where one puts back the cards in the deck as they are removed; but it will be more exact to multiply these squares by the ratio $(s+1)/s = 313/312$, as we have done previously (no. 7), in calculating the probability of the refait of 31. Therefore in calling q the probability of any null coup, we will have

$$q = (p_2^2 + p_3^2 + p_4^2 + \cdots + p_{10}^2)\frac{s+1}{s};$$

which gives, in carrying out the numerical calculation,

$$q = 0.08810.$$

12. In a long sequence of coups, the events occur, very nearly, proportionally to their respective probabilities. Thus, the ratio of the number of null coups to the total number of coups will deviate little from that of 881 to 10,000; and according to the probability of a refait of 31, as we have found (no. 7), the ratio of the number of refaits to the number of coups played, including the null coups, will be, very nearly, equal to that of 21,967 to 1,000,000. But these proportions are the limits which the results of chance should approach indefinitely, as the number of coups will become greater; and one can determine, for each number of coups, the probability that these results will not deviate from their limits beyond a given quantity.

In designating by n the total number of coups, by m that of the refaits of 31, by p the probability of a refait, by r the probability that the difference $m/n - p$ will be included between the two limits given and represented by

$$-\frac{t\sqrt{2p(1-p)}}{\sqrt{n}}, \qquad \text{and} \qquad \frac{t\sqrt{2p(1-p)}}{\sqrt{n}},$$

one will find

$$r = \frac{2}{\sqrt{\pi}}\int e^{-t^2}\,dt + \frac{e^{-t^2}}{\sqrt{2p(1-p)n\pi}}; \qquad (*)$$

the integral beginning with t,[1] e being the base of the Naperian logarithms, and π the ratio of the circumference to the diameter. When the variable t will increase, the probability r will approach unity or certitude; but, at the same time, the limits of the difference $m/n - p$ will be more extensive. On the other hand, when t will diminish, these limits will

(*) Analytical Theory of Probabilities, page 280.

[1] Translator's note: The interval of integration is $[0, t]$.

be narrower; but the probability r which corresponds to them will weaken itself and decay by becoming very small and little different from

$$\frac{1}{\sqrt{2p(1-p)n\pi}}.$$

When it will be equal to $\frac{1}{2}$, one will be able indifferently to expect that this difference $m/n - p$ will fall within or without these limits. The variable t remaining the same, if the number n increases indefinitely, the probability r will vary very little, and the limits of the difference $m/n - p$ will decrease more and more; so that one can always take n sufficiently large that these limits fall below any given quantity. If one would consider the difference $m - np$ between the number m of the refaits observed, and the number np of the refaits calculated according to their probability p, the limits of this difference would be

$$\pm t\sqrt{2p(1-p)n};$$

the probability r being the same as before. These limits will extend farther and farther, as the number n increases, but they will increase less rapidly than that number, and only in the ratio of its square root.

In putting for p the value 0.021967, the limits of the difference $m - np$ will become[2]

$$\pm(0.2074)t\sqrt{n}.$$

If one has, for example, $n = 1{,}000{,}000$, and one takes $t = 3$, there will be the probability r that the number m of refaits observed will be included between

$$21{,}967 - 622 \qquad \text{and} \qquad 21{,}967 + 622.$$

In order to calculate the value of r, one will first take the integral that this expression contains, from $t = 0$ to $t = \infty$; which gives

$$\int e^{-t^2}\,dt = \frac{1}{2}\sqrt{\pi};$$

then one will deduct from this integral its value taken from $t = 3$ to $t = \infty$, value which will be given by the series

$$\int e^{-t^2}\,dt = e^{-t^2}\left(\frac{1}{2t} - \frac{1\cdot 3}{2^2t^3} + \frac{1\cdot 3\cdot 5}{2^3t^5} - \frac{1\cdot 3\cdot 5\cdot 7}{2^4t^7} + \cdots\right),$$

in which one will make $t = 3$. One finds, in this manner,

$$r = 0.9998;$$

[2] Translator's note: Poisson had 0.2674 in place of 0.2074.

so that there is nearly 5,000 to bet against 1 that, in a million coups, the number of refaits will not be less than 21,345, and will not exceed 22,589. If the number observed departs from these limits, it would be very probable that there was an unintentional error, or that someone cheated at the game.

One can wonder what should be the value of t for which one would have $r = \frac{1}{2}$: this value will be given by the equation

$$t - \frac{t^3}{1 \cdot 3} + \frac{t^5}{1 \cdot 2 \cdot 5} - \frac{t^7}{1 \cdot 2 \cdot 3 \cdot 7} + \frac{t^9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 9} - \cdots = \frac{\sqrt{\pi}}{4} - \frac{e^{-t^2}}{2(0.2074)\sqrt{n}}.$$

In neglecting first the second term of its second part, one finds that the value of t is very nearly equal to 0.45. If one then makes $t = 0.45 + x$, and one determines x, by neglecting its square and its higher powers, one finds

$$t = 0.4460 - \frac{1}{2(0.2074)\sqrt{n}};$$

from which it results that there is one against one to bet that in a large number n of coups, the number m of refaits will be included between the two limits

$$(0.021967)n \pm \left[(0.0924)\sqrt{n} - \frac{1}{2}\right].$$

Thus, in a million coups, for example, it will be indifferent to bet that the number of refaits will differ from 21,967, more or less, by a number greater or less than 92.

In general, when two players play one against the other, in a fair game, there is the probability r that the number of trials that one of the two, without designating which, will win more than the other, in a very great number of coups, will not exceed the double of $t\sqrt{2p(1-p)n}$, while making in this double $p = \frac{1}{2}$; which gives $t\sqrt{2n}$. If therefore one takes for t the value which corresponds to $r = \frac{1}{2}$, there will be one against one to bet that the difference between the numbers of trials won by the two players will not exceed

$$(0.6307)\sqrt{n} - 1;$$

which makes 630, for each million trials. There will therefore be a disadvantage to bet, for example, that one of the players would win fewer than 600 trials more than the other, and an advantage to bet that the difference of the trials won will not exceed 650 trials.