

Math 431 Homework 5  
Due 10/23

For the purposes of this homework I will list a few lemmas here and use them. You may want to prove this as an exercise for your own well being.

Betweenness Lemmas

**Lemma 1.** Let  $A$  and  $B$  be two distinct points.

- (1)  $AB = BA$ .
- (2)  $AB \subset \overrightarrow{AB} \subset \overleftarrow{AB}$ .

**Lemma 2.** If  $A * B * C$  and  $l$  is any line passing through  $C$  that is distinct from  $\overleftarrow{AC}$ , then  $A$  and  $B$  are on the same side of  $l$ .

**Lemma 3.** If  $A$  and  $B$  are distinct points on the same side of a line  $l$  and  $\overleftarrow{AB}$  intersects  $l$  at a point  $C$ , then  $A * B * C$  or  $B * A * C$ .

**Lemma 4.** If  $A * B * C$  and  $l$  is a line passing through  $B$  that is distinct from  $\overleftarrow{AC}$ , then  $A$  and  $C$  are on opposite sides of  $l$ .

**Lemma 5.** If  $A$  and  $C$  are on opposite sides of a line  $l$ , then there exists a unique point  $B$  such that  $l$  passes through  $B$  and  $A * B * C$ .

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1. Prove the second part of Proposition 3.3. That is, given  $A * B * C$  and  $A * C * D$ , prove that  $A * B * D$  (do not use the first part of the same proposition. Never mind! You may. But if you have the solution that does not use it, you'll get extra credit). Follow the book's suggestion by considering the line  $\overleftrightarrow{EB}$ . Is the converse of Proposition 3.3 true?

*Proof.* From the proof of the first part we know that  $A, B, C, D$  are distinct collinear points and  $B * C * D$ ; we also have a point  $E$  not lying on line  $\overleftrightarrow{AD}$ . By Lemma 2,  $A$  and  $C$  are on opposite sides of  $\overleftrightarrow{EB}$ . By Lemma 4,  $C$  and  $D$  are on the same side of  $\overleftrightarrow{EB}$ . Thus, by B-4(iii),  $A$  and  $D$  lie on opposite sides of  $\overleftrightarrow{EB}$ . By Lemma 3, there is a unique point lying on both  $\overleftrightarrow{AD}$  and  $\overleftrightarrow{EB}$ ; since  $B$  lies on both of these lines, this unique point is  $B$ . Therefore, the second part of Lemma 3 gives  $A * B * D$ .

The converse is true as well. We wish to show that  $B * C * D$  and  $A * B * D$  imply  $A * B * C$  and  $A * C * D$ .  $B * C * D$  and  $A * B * D$  can, by B1, be rewritten as  $D * C * B$  and  $D * B * A$ . Consider the following mapping of letters:  $D \rightarrow A', C \rightarrow B', B \rightarrow C', A \rightarrow D'$ . Our claim now reads:  $A' * B' * C'$  and  $A' * C' * D'$  imply  $B' * C' * D'$  and  $A' * B' * D'$ , which is exactly the statement of Proposition 3.3. and we have already supplied its proof. □

2. Prove Proposition 3.5. If  $A * B * C$ , then  $AC = AB \cup BC$  and  $B$  is the only point in common segments  $AB$  and  $BC$ .

*Proof.* (1) First let us prove that  $AB \cup BC \subset AC$ . We split this into two parts: first we show  $AB \subset AC$ , and then we show  $BC \subset AC$ .

Suppose that  $P \in AB$ . If  $P = A$  then  $P \in AC$ . If  $P = B$ , then substituting  $B$  for  $P$  in  $A * B * C$  gives  $A * P * C$ , so  $P \in AC$ . Now suppose that  $P \neq A$  and  $P \neq B$ . Since  $P \in AB$ , we have  $A * P * B$ . Combining this with  $A * B * C$  gives  $A * P * C$  by Proposition 3.3. Thus,  $P \in AC$ . We conclude that  $AB \subset AC$ .

We have  $C * B * A$  by hypothesis and B-1. The result of the previous paragraph allows us to conclude that  $CB \subset CA$ . By Lemma 1(a),  $BC \subset AC$ .

Having proved that  $AB \subset AC$  and  $BC \subset AC$ , we conclude that  $AB \cup BC \subset AC$ .

(2) Let us prove that  $AC \subset AB \cup BC$ . Suppose that  $P \in AC$ . The goal is to prove that  $P \in AB$  or  $P \in BC$ . If  $P = A$  or  $P = B$  then  $P \in AB$ , and if  $P = C$  then  $P \in BC$ . Let us now assume that  $P, A, B, C$  are distinct. Since  $P \in AC$ , we have  $A * P * C$ . Note that  $P, A, B, C$  are collinear by the argument from Exercise 1 on Problem Set 4. Let  $l$  be a line through  $P$  that is different from the line through  $P, A, B, C$ . Such a line exists by Proposition 2.3 and I-1.

Since  $P$  is on  $l$ , by Proposition 2.1,  $A$  does not lie on  $l$  and neither does  $B$ . Thus, either  $B$  and  $A$  are on the same side of  $l$ , or  $B$  and  $A$  are on opposite sides of  $l$ . Before dealing with each case, let us remark that by Lemma 2,  $A$  and  $C$  are on opposite sides of  $l$ .

Suppose that  $B$  and  $A$  are on the same side of  $l$ . By B-4(iii)  $B$  and  $C$  are on opposite sides of  $l$ . As  $P$  is the unique point common to both  $l$  and  $\overleftrightarrow{AC}$  (Proposition 2.1), Lemma 3 gives  $B * P * C$ . So  $P \in BC$ .

Suppose that  $B$  and  $A$  are on opposite sides of  $l$ . By B-4(ii)  $B$  and  $C$  are on the same side of  $l$ . By Lemma 5,  $B * C * P$  or  $C * B * P$ . The first option,  $B * C * P$ , together with  $A * B * C$  (hypothesis) gives  $A * C * P$  by Proposition 3.3, Corollary 1. This contradicts  $A * P * C$  by B-3. So  $B * C * P$  cannot hold. We are left with the case  $C * B * P$ . Combining this with  $A * P * C$  gives  $A * P * B$  by Proposition 3.3 Corollary 1. Thus  $P \in AB$ . This completes the proof.

(3)

Certainly  $B$  is in both segments  $AB$  and  $BC$ . We claim that there is not other point in both segments. Suppose to the contrary that there is such a point  $P \neq B$  such that  $P \in AB$  and  $P \in BC$ . Points  $A, B$ , and  $C$  are distinct by B-1. Note that  $P \neq A$  because if it were, then  $A$  would be in  $BC$ ; so  $B * A * C$ , which contradicts  $A * B * C$  (hypothesis) by B-3. By the same logic,  $P \neq C$ .

Let  $E$  be a point not lying on  $\overleftrightarrow{AC}$ . Consider the line  $\overleftrightarrow{EB}$ . Since  $P \in AB$ , we have  $A * P * B$ , so by Lemma 4,  $A$  and  $P$  are on the same side of  $\overleftrightarrow{EB}$ . Since  $P \in BC$ , we have  $B * P * C$ , so by Lemma 4,  $P$  and  $C$  are on the same side of  $\overleftrightarrow{EB}$ . By B-4(i),  $A$  and  $C$  are on same side of  $\overleftrightarrow{EB}$ . But by Lemma 2,  $A$  and  $C$  are on opposite sides of  $\overleftrightarrow{EB}$ , a contradiction. Thus  $P = B$ , and this is the only point that lies on  $AB$  and  $BC$ .  $\square$

**3.** Prove Proposition 3.6: If  $A * B * C$  then  $B$  is the only point common to both rays  $\overrightarrow{BA}$  and  $\overrightarrow{BC}$ , and  $\overrightarrow{AB} = \overrightarrow{AC}$ .

*Proof.* (1) Suppose  $P$  is a point in  $\overrightarrow{BA}$  and  $\overrightarrow{BC}$ . We want to show that  $P = B$ . Since  $P \in \overrightarrow{BA}$ , we have  $P \in BA$  or  $B * A * P$ .

**Case 1:** Suppose that  $P \in BA$ . Since  $P \in \overrightarrow{BC}$ , we have  $P \in BC$  or  $B * C * P$ . Let us consider these two subcases:

**Case 1a:** Suppose  $P \in BC$ . Then  $P = B$  by the uniqueness part of Proposition 3.5.

**Case 1b:** Suppose  $B * C * P$ . Then  $P \neq B$  and  $P \neq C$ . Since  $P \in BA$ , we have  $B * P * A$ . By Proposition 3.3,  $B * C * A$ , which contradicts  $A * B * C$  by B-3.

**Case 2:** Suppose that  $B * A * P$ . Again, since  $P \in \overrightarrow{BC}$  we have two subcases:

**Case 2a:** Suppose  $P \in BC$ . We know  $P \neq B$  by B-1. Thus  $B = C$  or  $B * P * C$ . If  $P = C$  then  $B * A * C$ . But  $A * B * C$  by assumption. This contradicts B-3. If  $B * P * C$ , then  $A * B * P$ . But  $B * A * P$  by assumption. This contradicts B-3.

**Case 2b:** Suppose  $B * C * P$ . Combining this with  $A * B * C$  gives  $A * B * P$  by Proposition 3.3, Corollary 1. This contradicts  $B * A * P$  by B-3.

This completes the case-by-case argument that  $P = B$ . Thus,  $B$  is the only point in  $\overrightarrow{BA} \cap \overrightarrow{BC}$ .

(2) We will show that  $\overrightarrow{AB} \subset \overrightarrow{AC}$ . Let  $P \in \overrightarrow{AB}$ . The goal is to show that  $P \in \overrightarrow{AC}$ . Given that  $P \in \overrightarrow{AB}$ , we have  $P = A$  or  $P = B$  or  $A * P * B$  or  $A * B * P$ . If  $P = A$  then  $P \in \overrightarrow{AC}$ . If  $P = B$ , then substituting  $P$  for  $B$  in  $A * B * C$  gives  $A * P * C$ , so  $P \in \overrightarrow{AC}$ . If  $A * P * B$ , then together with  $A * B * C$ , we get  $A * P * C$  by Proposition 3.3, so  $P \in \overrightarrow{AC}$ . We are left with one more case to consider:  $A * B * P$ .

Suppose that  $A * B * P$ . Consider the relations among points  $P, A, C$ . If  $P = C$  then  $P \in \overrightarrow{AC}$ . If  $P \neq C$  then  $P, A, C$  are distinct collinear points ( $P \neq A$  because  $A * B * P$ , and  $A \neq C$  because  $A * B * C$ ). By B-3,  $P * A * C$  or  $A * P * C$  or  $A * C * P$ . Either of the last two cases implies  $P \in \overrightarrow{AC}$ . The first case,  $P * A * C$ , combined with  $A * B * C$  gives  $P * A * B$  by Proposition 3.3 Corollary 1; this contradicts  $A * B * P$  by B-3.

(3) Finally, we will show that  $\overrightarrow{AC} \subset \overrightarrow{AB}$ . Let  $P \in \overrightarrow{AC}$ . Then  $P = A$  or  $P = C$  or  $A * P * C$  or  $A * C * P$ . If  $P = A$  then  $P \in \overrightarrow{AB}$ . If  $P = C$ , then  $A * B * P$ , so  $P \in \overrightarrow{AB}$ . If  $A * C * P$  then  $A * B * P$  by Proposition 3.3, so  $P \in \overrightarrow{AB}$ . We are left with one more case to consider:  $A * P * C$ .

Suppose that  $A * P * C$ . Consider the relations among the points  $P, B, A$ . If  $P = B$  then  $P \in \overrightarrow{AB}$ . If  $P \neq B$ , then  $P, A, B$  are distinct collinear points. By B-3,  $P * A * B$  or  $A * P * B$  or  $A * B * P$ . Either of the last two cases implies  $P \in \overrightarrow{AB}$ . The first case,  $P * A * B$ , combined with  $A * B * C$ , gives  $P * A * C$ , contradicting  $A * P * C$  by B-3. Therefore,  $\overrightarrow{AC} \subset \overrightarrow{AB}$ .  $\square$