Class #7

Interpretations

Proposition 2.5: For every point P there exist at least two lines through P.

Proof #1: Let P be a point. By Proposition 2.4 there is a line *l* that is not incident with P. By axiom ***I-2*** there are at least two distinct points, S and T, incident with *l*. Since P does not lie on *l*, P can not equal S or T. By applying axiom ***I-1*** twice we conclude that there exists a line *m* that passes through P and S and that there exists a line *n* that passes through P and T. *m* and *n* both pass through P and they are not equal to each other (if they were they would both pass through S and T, and would consequently have to equal *l*, by axiom ***I-1***, which they can not as P lies on each of them, but does not lie on *l*).

Proposition 2.5: For every point P there exist at least two lines through P.

Proof #2: Let P be a point. By axiom ***I-3*** there exist three distinct points R, S and T. There are two possibilities:

1. $P \in \{R, S, T\}$
2. $P \notin \{R, S, T\}$

Suppose $P \in \{R, S, T\}$. Then P equals one of these points, say R. However, $P \neq S$ and by ***I-1***, there is a unique line *l* through P and S. Also, $P \neq T$, so by ***I-1***, there is a unique line *m* through P and T. Since no line passes through all three of the points R, S and T, we conclude that *l* and *m* are distinct, and we are done.

Suppose $P \notin \{R, S, T\}$. Since $P \neq R$ and by axiom ***I-1***, there is a unique line *l* through P and R. If S does not lie on *l*, then the unique line *m* passing through P and S whose existence is guaranteed by axiom ***I-1*** is not equal to *l* and our claim is proven. If, however, S lies on *l*, then T does not lie on *l* (because of the choice of R, S and T), and there is a line *n* through P and T, by axiom ***I-1***. Since T lies on *n* and not on *l*, $l \neq n$.

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If H then C ($H \Rightarrow C$)

- CONTRAPOSITIVE
  - If not C then not H ($\sim C \Rightarrow \sim H$)
  - Logically equivalent to $H \Rightarrow C$
- CONVERSE
  - If C then H ($C \Rightarrow H$)

Exercise: State the converse and contrapositive of Proposition 2.1.

- Proposition 2.1: If *l* and *m* are distinct lines that are not parallel, then *l* and *m* have a unique point in common.

- Which one of the two can you prove?

Food for thought

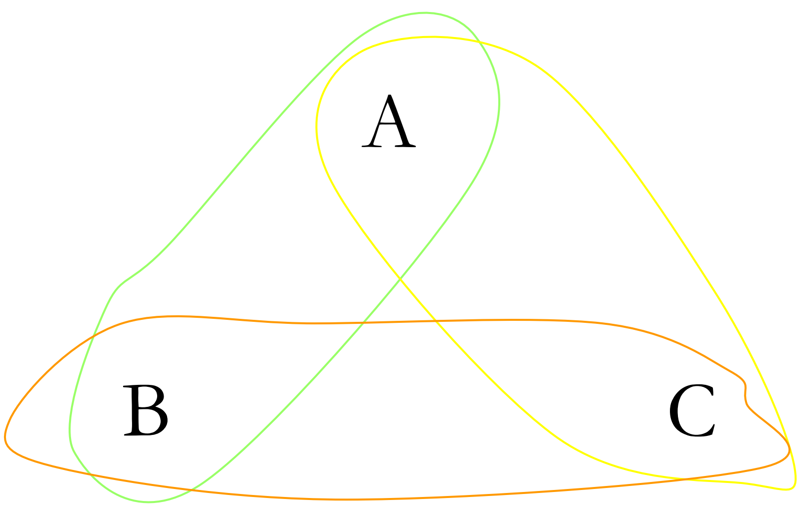
- Is it possible to prove one of the axioms of incidence geometry from the other two?
- How do you convince somebody that it is not possible to prove a certain statement from a given list of axioms?

Interpretation

- If we give the undefined terms of a system a particular meaning then we have given that system an *interpretation*.

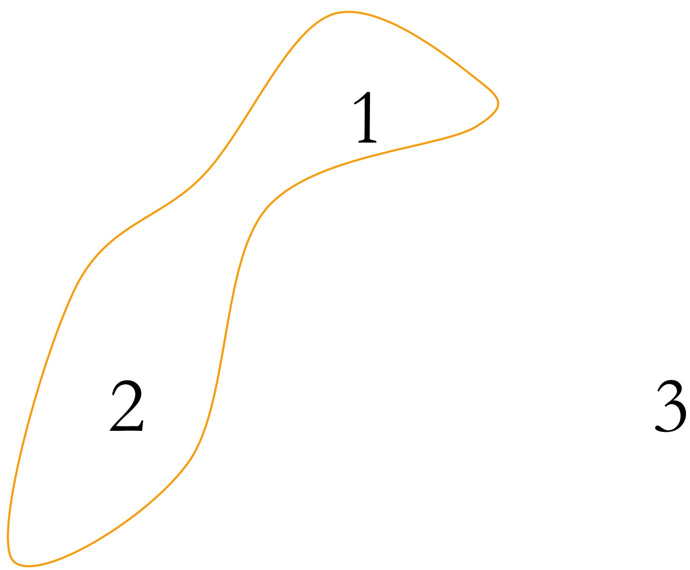
Interpretation of incidence geometry (IG) #1

- Points – letters A, B, and C
- Lines – sets {A,B}, {A,C} and {B,C}
- Point lies on a line *l* – the letter belongs to the set *l*



Interpretation of IG #2

- Points – numbers 1, 2, and 3
- Lines – there is only one line *l*
- 1 lies on *l*, 2 lies on *l*, 3 does not lie on *l*



Models

- Q: Are the axioms of incidence geometry correct statements in these interpretations?
- If an axiom is a correct statement in a given interpretation we say that that interpretation satisfies the axiom
- If an interpretation satisfies all the axioms of a given system we say it is a model for that system.

Exercise 1:

1. Is interpretation #1 of IG a model of IG?
    1. I1 …
    2. I2 …
    3. I3 …
2. Is interpretation #2 of IG a model of IG?
    1. I1 …
    2. I2 …
    3. I3 …

- To demonstrate that a statement S can not be proved from a list of statements $\mathcal{L}$ it is enough to find an interpretation in which all the statements $\mathcal{L}$ are correct, but S is not.
- WARNING: If you find an interpretation in which S is correct, that does not mean that there is a proof of S.