Language: logical terms

- A *statement* is a sentence that is either true or false, but not ambiguous

Are the following statements?

1. The temperature in Ann Arbor at 7:30am on 9/8/2006 is 73F.
2. Bicycles have nine wheels.
3. The 36th digit of π is 7.
4. Is it hot today?
5. Mona Lisa is a beautiful painting.
6. She is 5 feet 1.

"She is 5 feet 1"

- Replace "she" by
  - Emina: Emina is 5 feet 1.
  - Michael Jordan: Michael Jordan is 5 feet 1.
- "she" is a free variable: it can take on different values
- A sentence with a free variable in it that becomes a statement when the free variable takes on a particular value is called a *predicate*.

Examples of predicates

1. $x + y = 3$
2. She went to the movies and he went to feed their dogs.

Notation:

1. P(x,y):="x + y = 3"
2. Q(she, he):="She went to the…."

Exercise

- Give examples of mathematical predicates that have 2 and 3 free variables. Share with your group.

Promoting predicates into statements

1. Substituting a particular value for the free variable

   P(7,3): =“ 7+3=3 ”

2. Giving a range of values for the free variable that turn the predicate into a statement

   T(x): =“ $x^2 - 1 = 0$”

- For all real numbers x, $x^2 - 1 = 0$
- There exists a real number x, $x^2 - 1 = 0$

False statement

True statement

Quantifiers

- *For all* ,∀, and *there exists*, ∃, are called quantifiers.
- Variables are no longer free; we now call them *bound*.

Exercises

1. Free variable z will refer to fish. Give examples of predicates A(z) and B(z) so that
   1. "For all z, A(z)" is a true statement
   2. "For all z, B(z)" is false, but "There exists z, B(z)" is true.

2. Translate the following statement into a precise statement
   1. "A line must pass through at least two points."

3. Consider the following statements
   1. For all x there exists y such that $y^2=x$.
   2. There exists y such that for all x, $y^2=x$.
   3. There exists x and there exists y such that $y^2=x$.
   4. There exists y and there exists x such that $y^2=x$.

Explain the differences between 1. and 2. and a. and b, if there are any. Which of the above statements are true?

Compound statements

P	Q	$P \wedge Q$ P and Q	$P \vee Q$ P or Q	$P \Rightarrow Q$ P implies Q	~P Not P
T	T	T	T	T	F
T	F	F	T	F	F
F	T	F	T	T	T
F	F	F	F	T	T

- "P only if Q" means "if P then Q"
- "P if and only if Q" means "(if P then Q) and (if Q then P)"
  - If and only if is abbreviated iff.
- "A=B" means that A and B represent identical object (eg. point).

Mathematical implication

- All mathematical statements are of this form (even when it does not appear to be so)

  - If (hypothesis) then (conclusion).
- Theorem: Base angles of an isosceles triangle are congruent.

If a triangle is isosceles, then its base angles are congruent.

Example

- If the moon is made of green cheese, then chocolate prevents cavities.
  - P:= "the moon is made of green cheese" is false
  - Q:= "chocolate prevents cavity" is false
  - $P \Rightarrow Q$ is true!

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Exercise

- Find the truth table for "Q ∨ ~P"

P	Q	~P	Q ∨ ~P
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

P ⇒ Q
T
F
T
T

Two statements whose truth tables are the same are called *logically equivalent*

Find the truth table for $P \vee \sim P$ and $P \wedge \sim P$

P	~P	$P \vee \sim P$	$P \wedge \sim P$
T	F	T	F
F	T	T	F

A statement whose truth table value is always true is called *tautology*.

A statement whose truth table value is always false is called *contradiction*.

Negation

- Q: What is the negation of:
  - For all x, P(x)
  - There exists y such that Q(y)