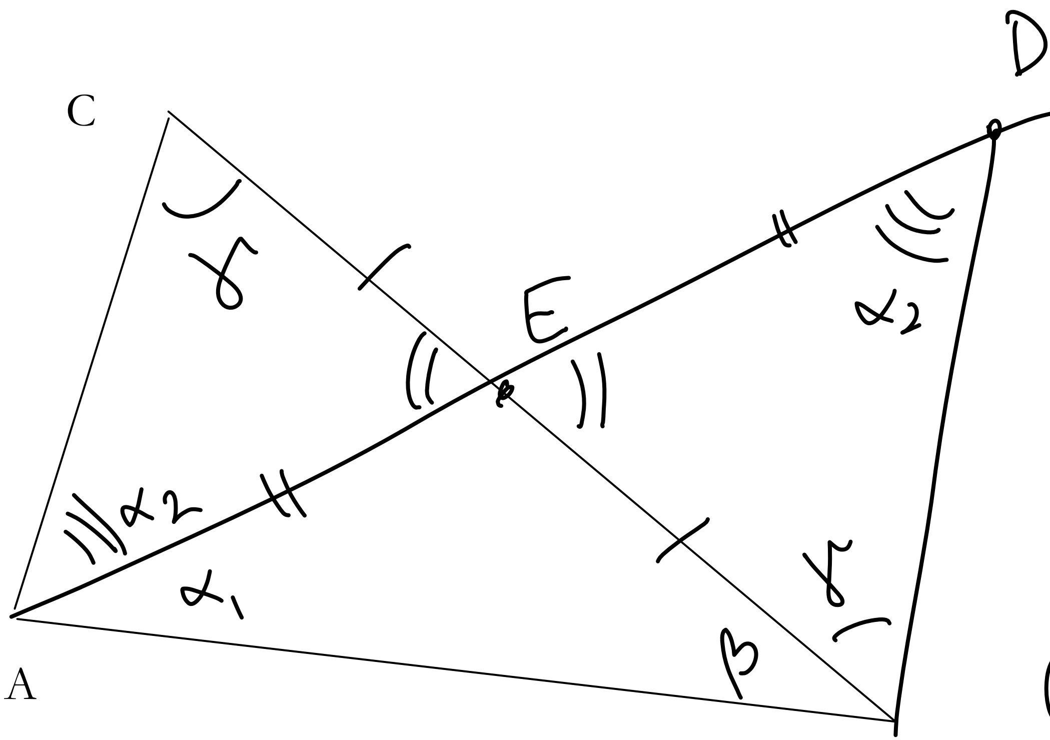
Theorem 5.3: Angle sum of any triangle is less than or equal to 180°

- Suppose there is a triangle with angle sum greater than 180°, say angle sum of ΔABC is 180° + p, where $p>0$.
- Goal: Construct a triangle that has the same angle sum, but one of its angles is smaller than p.
  - Why is that enough?
  - We would have that the remaining two angles add up to more than 180°, which contradicts one of our theorem HW9#3.

Construct a triangle with angle sum as that of ΔABC ($180^\circ + p$), but one of its angles is at most half of $m(\sphericalangle A)$



Since $\alpha_1 + \alpha_2 = m(\sphericalangle A)$, we conclude that either α_1 or α_2 is smaller than $m(\sphericalangle A)$, so either $m(\sphericalangle DAB)$ or $m(\sphericalangle ADB) \leq m(\sphericalangle A)/2$

$\Delta ECA \cong \Delta EBD$ by SAS

$\Rightarrow \sphericalangle ACE \cong \sphericalangle DBE$

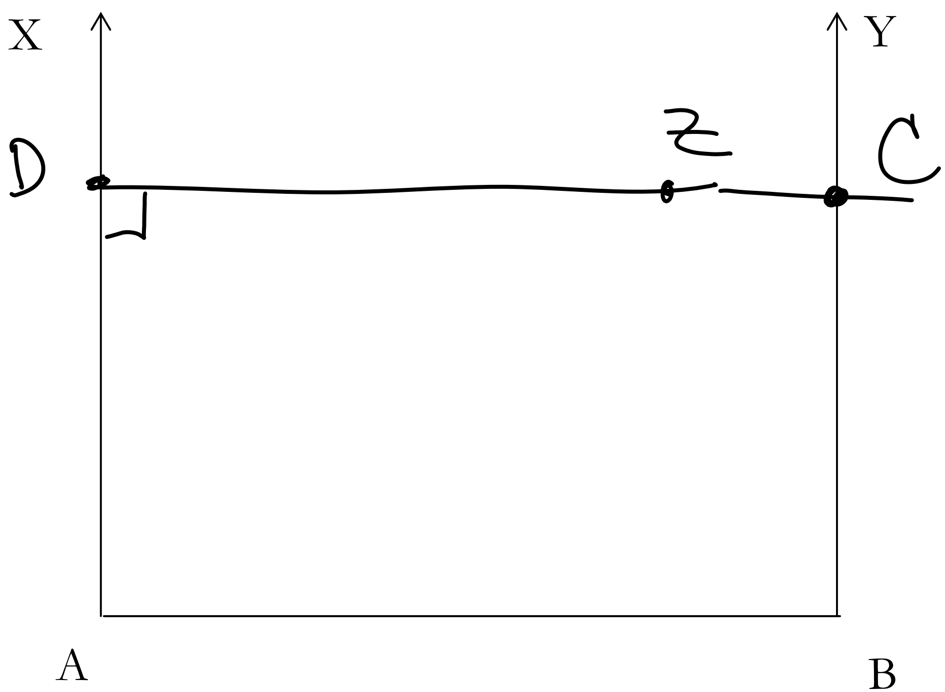
$\sphericalangle CAE \cong \sphericalangle EDB$

Angle sum of ΔABC is $\alpha_1 + \alpha_2 + \beta + \gamma = 180^\circ + p$

Angle sum of ΔABD is $\alpha_1 + \alpha_2 + \beta + \gamma = 180^\circ + p$

If EPP holds a rectangle exists.

Let $\overleftrightarrow{AX}$ and $\overleftrightarrow{BY}$ be perpendicular lines to $\overleftrightarrow{AB}$ such that X and Y are on the same side of $\overleftrightarrow{AB}$



Let D be any point on ray $\overrightarrow{AX}$. There exists a unique ray $\overrightarrow{DZ}$ such that Z is on the same side of $\overleftrightarrow{AX}$ as B and that $\sphericalangle ADZ$ is a right .

Note that $\overleftrightarrow{DZ}$ is parallel to $\overleftrightarrow{AB}$ (AIA thm)

Line $\overleftrightarrow{DZ}$ intersects $\overleftrightarrow{BY}$ in a point C, for if it did not then $\overleftrightarrow{AB}$ and $\overleftrightarrow{BY}$ are two distinct lines parallel to $\overleftrightarrow{DZ}$ through B, which contradicts EPP.

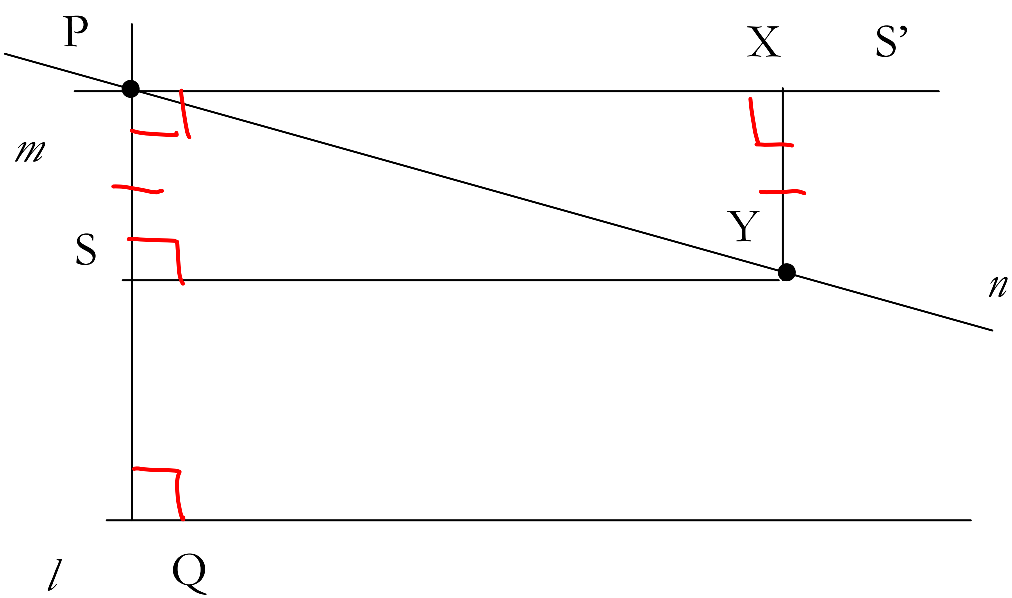
C is on the same side of $\overleftrightarrow{AB}$ as D because Note.

Using converse of alternate interior angle theorem we conclude that $\sphericalangle YCB \cong \sphericalangle ABC$, where D*C*Y. Therefore $\sphericalangle YCB$ is a right angle. Its supplement is $\sphericalangle DCB$, hence it is a right angle as well

If a rectangle exists then EPP holds.

Let l be any line and P a point not lying on it. Let Q be the foot of the perpendicular to l through P. Let m be a line perpendicular to $\overleftrightarrow{PQ}$ through P. Then m parallel to l. Claim is that every line n through P not equal to m intersects l.

If $n=\overleftrightarrow{PQ}$ then n intersects l. Suppose then Q is not on n.



A ray $\overrightarrow{PY}$ of n lies between ray $\overrightarrow{PQ}$ and a ray $\overrightarrow{PS'}$ of m.

Let X be the foot of the perpendicular to m through Y.

Let S be the foot of the perpendicular to PQ through Y. Then □SYXP has three right angles, so its fourth angle has to be right as well (rectangle exists implies every triangle has angle sum 180° implies every quadrilateral has angle sum 360°)

PS≅XY (still to show). We can choose Y so that XY>PQ. Then PS>PQ, so P*Q*S. Since S and Y on the same side of l, and P and S on opposite side of l, PY intersects l.