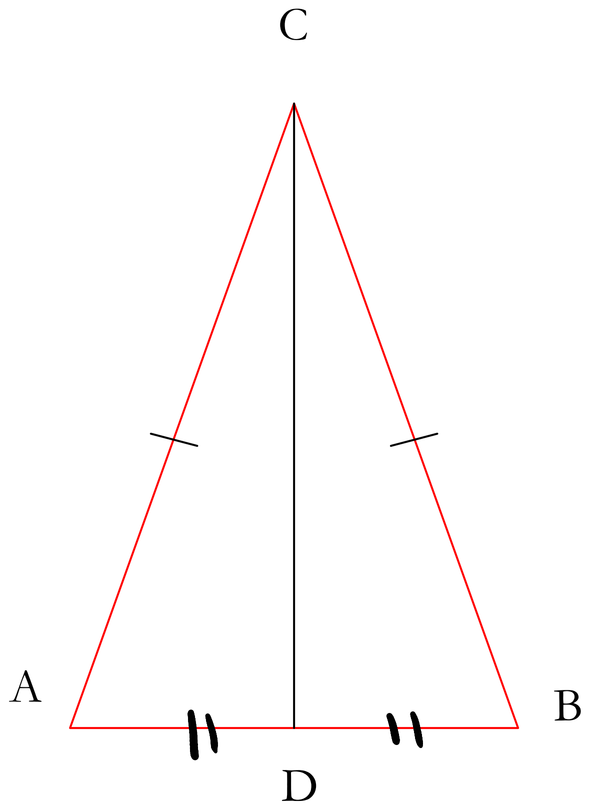
Class #28

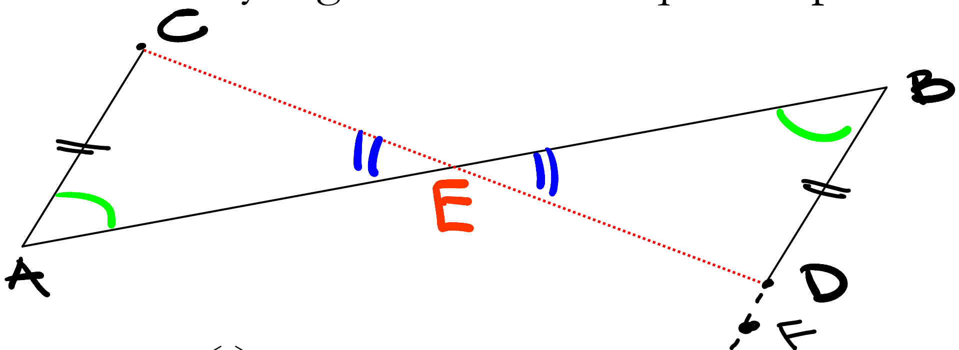
Midpoints, medians, bisectors

What do you think of this proof of "Base angles of isosceles triangle are congruent"?

Let ΔABC be a triangle with AC ≅ BC. Let D be a midpoint of AB. In triangles ΔACD and ΔBCD, AC ≅ BC by hypothesis. AD ≅ BD by definition of a midpoint. Therefore, triangles ΔACD and ΔBCD are congruent by SSS. Hence, ∢ A ≅ ∢ B.



Proposition 4.2.3 Every segment has a unique midpoint.



Proof: Let C be any point not on line $\overleftrightarrow{AB}$, whose existence is guaranteed by Prop. 2.3. By axiom C4 there is a unique ray $\overrightarrow{BF}$ on the opposite side of $\overleftrightarrow{AB}$ from C such that $\sphericalangle BAC \cong \sphericalangle ABF$ (in green). By C1 there is a unique point D on ray $\overrightarrow{BF}$ such that AC≅BD. We will first note that $\sphericalangle BAC$ and $\sphericalangle ABF$ are alternate interior angles cut by a transversal $\overleftrightarrow{AB}$ to lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$. Since these angles are congruent, we apply AIA theorem to conclude that $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are parallel lines. Since C and D are on opposite sides of $\overleftrightarrow{AB}$ we know that CD intersects the line $\overleftrightarrow{AB}$ at a point, call it E. Since E is on segment CD we have C*E*D. We also know that E lies on line $\overleftrightarrow{AB}$, so E could be either A or B, or if they are three distinct points we know what their relationship could be using B3. One of the following happens:

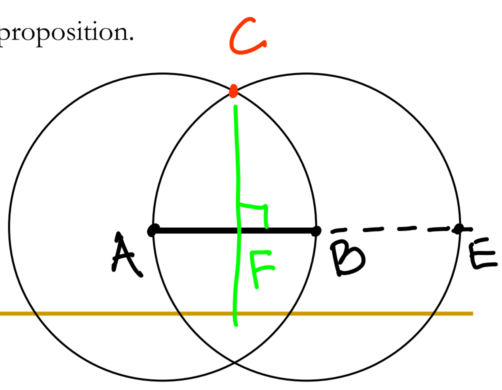
1. E=A : We have C*A*D, so D lies on both $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ which contradict the previous statement that these two lines are parallel.
2. E=B : As above.
3. E*A*B: By Lemma 3.2.4 E and B are on opposite sides of line $\overleftrightarrow{AC}$. Since line $\overleftrightarrow{BD}$ is parallel to $\overleftrightarrow{AC}$, BD does not intersect $\overleftrightarrow{AC}$, so B and D are on the same side of $\overleftrightarrow{AC}$. By B4 we have that E and D are on opposite sides of $\overleftrightarrow{AC}$. Since the intersection of lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{ED}$ is point C, by Lemma 3.2.5 we have E*C*D, which is a contradiction to B3 since we already have C*E*D.
4. A*B*E: Argument as in the previous case.
5. A*E*B: is the only remaining case.
6. Angles $\sphericalangle CEA$ and $\sphericalangle DEB$ are vertical, hence congruent, by Proposition 3.15. (in blue). If we had AAS (aka SAA), we could conclude that $\Delta AEC \cong \Delta BED$. By definition of congruent triangles we have AE ≅ EB. This together with A*E*B implies that E is a midpoint of AB.

Q: Why did I say "a midpoint" in the previous sentence? Should it be "the midpoint"?

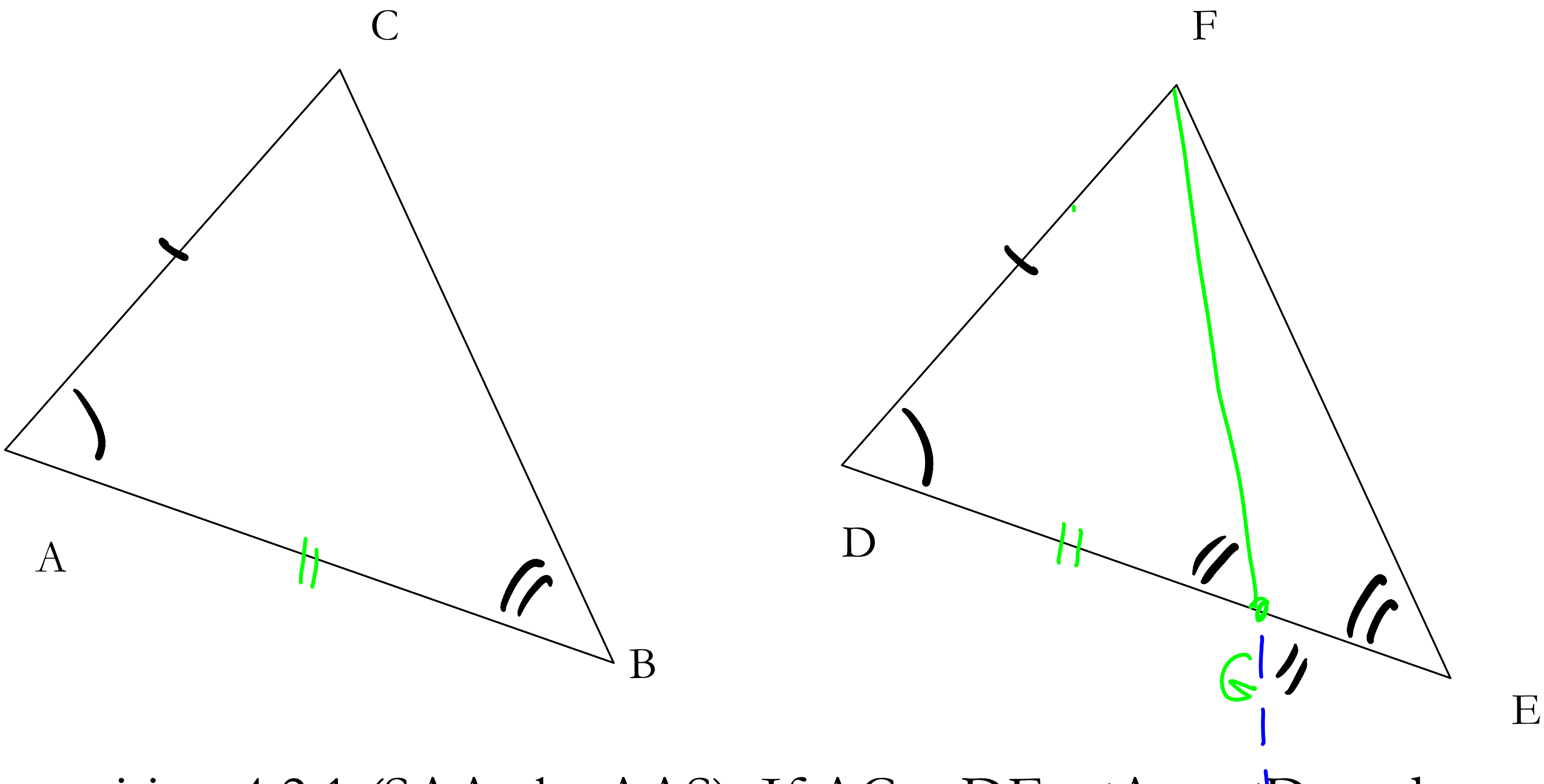
Suppose there are two distinct midpoints and use C1 to argue that those two points have to be equal.

Notes and remarks:

- **For 1:00 pm class**: Notice that in the previous argument there was no need to mess with the fact that the angle $\sphericalangle BAC$ is not a right angle It could be, but it doesn't have to be. It was not relevant in this argument at all.
- **Amanda, Matt & Edgar's method**: Their idea was to construct an isosceles triangle with a base AB. I have to admit that I still don't see how to resolve a problem we ran into today in class, so I am proposing a slightly different approach.
  - Let c be the circle with center A and radius AB, and let c_1 be the circle with center B and radius BA (congruent radii). Note that A lies on the circle c_1 and is inside the circle c. By C1 there is a unique point E on the ray opposite to $\overrightarrow{BA}$ such that BE ≅ BA. By definition of c_1, E is on the circle c_1. Since A*B*E, AB<AE, so E is outside the circle c. We now have a two points on c_1 one inside c one outside. By circular continuity principle, these two circles intersect in two points. Call one of them C. Since C is on both circles and those two circle have congruent radii, we have AC ≅ BC and ΔABC is isosceles. By Proposition 3.10 we have $\sphericalangle A \cong \sphericalangle B$. Let F be the foot of the perpendicular to AB through C (we note that F is not equal to A or B, because if it were we would have that $\sphericalangle CAB$ and $\sphericalangle CBA$ would be right angles, hence congruent, and by AIA thm lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ would be parallel, but they both contain C). Then $\sphericalangle AFC$ and $\sphericalangle BFC$ are right angles, hence are congruent by Proposition 3.22. We now have two triangles with one pair of congruent sides and two pairs of congruent angles, so we can use AAS (aka SAA) to conclude that ΔAFC ≅ ΔBFC. By definition of congruent triangles AF ≅ FB. It remains to be argued that A*F*B. Since F, A and B are three distinct points either
    - F*A*B: By definition of <, we have FA < FB, but that contradicts ordering of segments proposition.
    - A*B*F: Similar to the above.
    - A*F*B: is the only remaining possibility, hence F is a midpoint of AB.



Need SAA



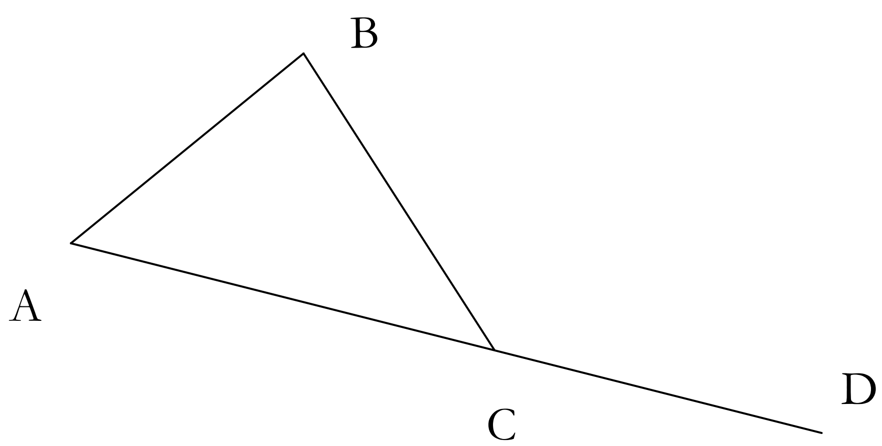
Proposition 4.2.1 (SAA aka AAS): If AC ≅ DF, ∢A ≅ ∢D, and ∢B ≅ ∢E, then ΔABC ≅ ΔDEF.

Proof of AAS-SAA

- If we knew that $AB \cong DE$, then we could use SAS to conclude that $\Delta ABC \cong \Delta DEF$. Suppose those two segments are not congruent. Then by Ordering of segments propositions we know that either $AB<DE$ or $DE<AB$.
- Suppose the former was the case. By definition of $<$, there is a point G such that $D*G*E$ and $AB \cong DG$. We now have that $\Delta ABC \cong \Delta DGF$, by SAS. By definition of congruent triangles, we have that $\sphericalangle CBA \cong \sphericalangle FGD$. Since $\sphericalangle CBA \cong \sphericalangle FED$, by hypothesis, using C5 we conclude that $\sphericalangle FED \cong \sphericalangle FGD$. We can split ways here.
  - Let M be such that $D*G*M$. Then angle $\sphericalangle EGM \cong \sphericalangle FGD$, since they are vertical angles (Proposition 3.15). We now have $\sphericalangle EGM \cong \sphericalangle FED$ which are alternate interior angles cut by a transversal DE to the lines FE and FG. These two lines should be parallel by AIA theorem, but that is clearly not true as F lies on each of them.
  - $\sphericalangle FGD$ is an exterior angle to the triangle ΔGEF, hence is greater that either remote angle by exterior angle theorem (follows), in particular it is greater that $\sphericalangle FED$. This contradicts ordering of angles proposition, as we have that these two angles are congruent as well.
- The argument is identical if we assume that $DE<AB$.

Need to know something about exterior angles

- An angle supplementary to an angle of a triangle is called *exterior angle* of the triangle.
- Two angles of a triangle that are not adjacent to this exterior angle are called *remote interior angles*.

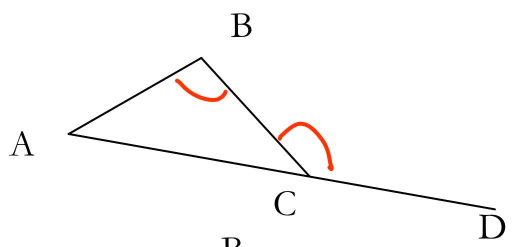


Exterior angle theorem

- Theorem 4.2: An exterior angle of a triangle is greater than either remote interior angle.
- Proof: Suppose contrary. Then either:
  1. $\sphericalangle DCB \cong \sphericalangle ABC$, or
  2. $\sphericalangle DCB < \sphericalangle ABC$.

Supply the arguments in each case:

1. We have



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