Class #26

Dedekind's axiom & Neutral Geometry

Dedekind's axiom

- Suppose that set $\{l\}$ of all points on a line l is a disjoint union $S_1 \cup S_2$ of two nonempty subsets so that no point of either subset is between two points of the other. Then there exists a unique point O on l such that one of the subsets is equal to a ray of l with vertex O and the other is equal to the rays complement.
- S_1 and S_2 are called *Dedekind's cut* of the line l.

Neutral geometry

- If we use I1-I3, B1-B4, C1-C6 and continuity axioms we can do a lot, but can not get Euclidean geometry. What we do get, we will call neutral geometry.

- Neutral geometry + Euclidean PP = Euclidean geometry

- Neutral geometry + Hyperbolic PP = Hyperbolic geometry

Alternate interior angles

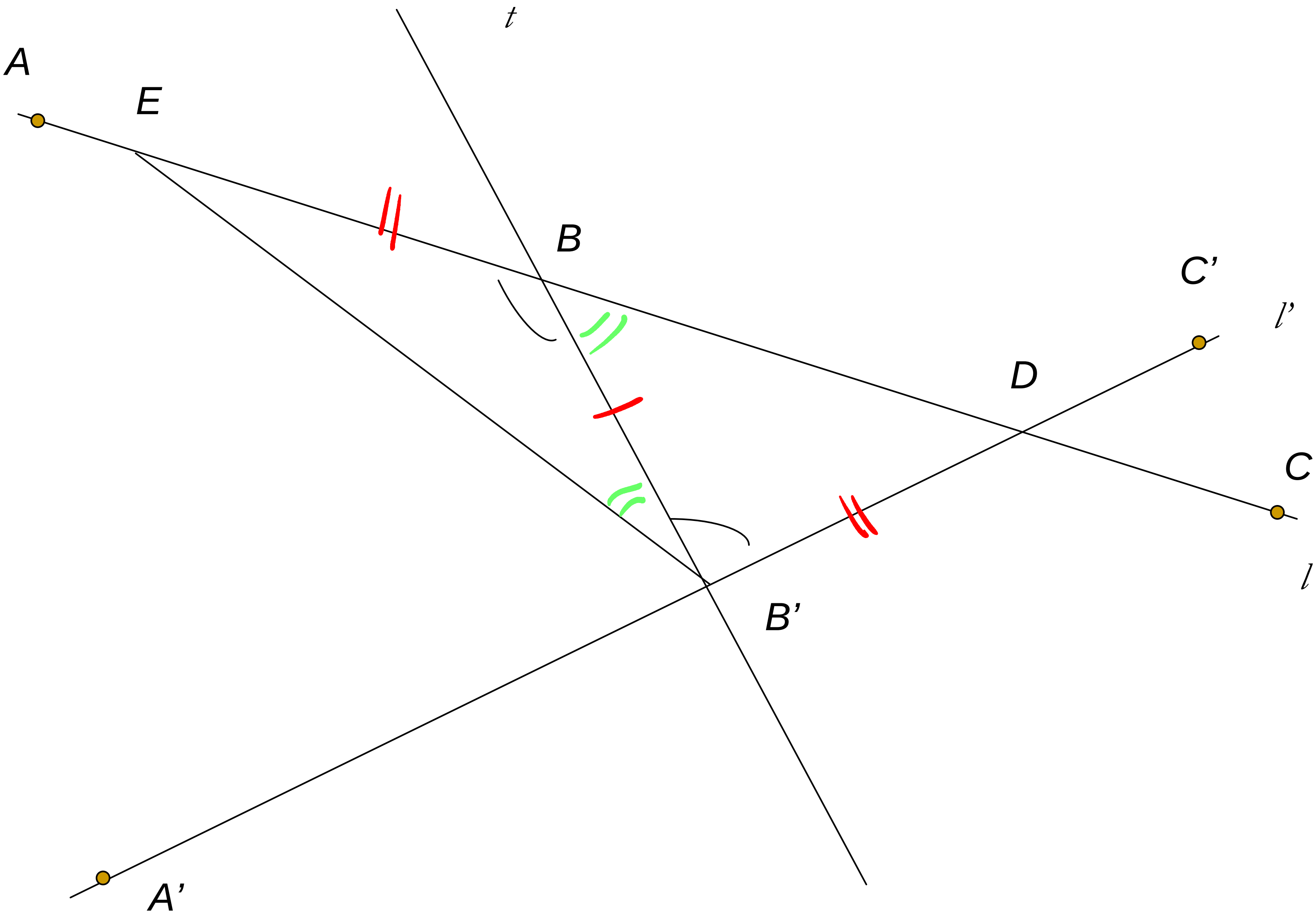
- We will say that a line t is a *transversal* to lines $l \neq l'$ if there are exactly two distinct points B and B' such that $\{l\} \cap \{t\} = \{B\}$ and $\{l'\} \cap \{t\} = \{B'\}$.
- Let A*B*C be points on l, A'*B'*C' be points on l' so that A and A' are on the same side of t. The angles ∢ABB', ∢CBB', ∢A'B'B and ∢C'B'B are called *interior*.
- Pairs (∢ABB',∢C'B'B) and (∢A'B'B,∢CBB') are called *alternate interior angles*.
- Let's see some pictures.

Alternate interior angle theorem

- If two lines cut by a transversal have a pair of congruent alternate interior angles, then the two lines are parallel.
- Proof: Let t be a transversal to lines l and l' and ∢ABB' ≅ ∢C'B'B (the notation is the same as in definition of alternate interior angles). Our claim is that the lines l and l' are parallel. Assume contrary, that is assume that they intersect, say at a point D. By axiom C1 there is a unique point E on the ray BA so that BE ≅ B'D. We now have (added in red in the figure)
  - BE ≅ B'D
  - B'B ≅ B'B
  - ∢EBB' ≅ ∢DB'B (hypothesis)

  so, by SAS, we conclude that ΔBB'D≅ΔB'BE. By definition of congruent triangles, we have ∢DBB' ≅ ∢BB'E (in green).

  By our hypothesis we know that ∢EBB' ≅ ∢DB'B. By Proposition 3.14 we know that congruent angles have congruent supplements, so the supplements of these two angels are congruent. Supplement of ∢EBB' is ∢DBB' , and let us call ∢DB'B's supplement ∢X, hence ∢X ≅ ∢DBB'. Angle ∢X and ∢EBB' both share a side, and they are congruent, so by Axiom C4 they have to be equal (that is their remaining sides have to coincide). Since ∢X = ∢EBB' is a supplement to ∢DB'B, we conclude that B'E and B'D are opposite rays, hence E and D lie on the same line, line l'. However, E and D also lie on l. We now have two lines l and l' passing through two distinct points, so by I1, $l = l'$, which contradicts our hypothesis. Therefore, our assumption must be wrong, and l and l' are in fact parallel.



t
A
E
B
C'
l'
D
C
l
B'
A'