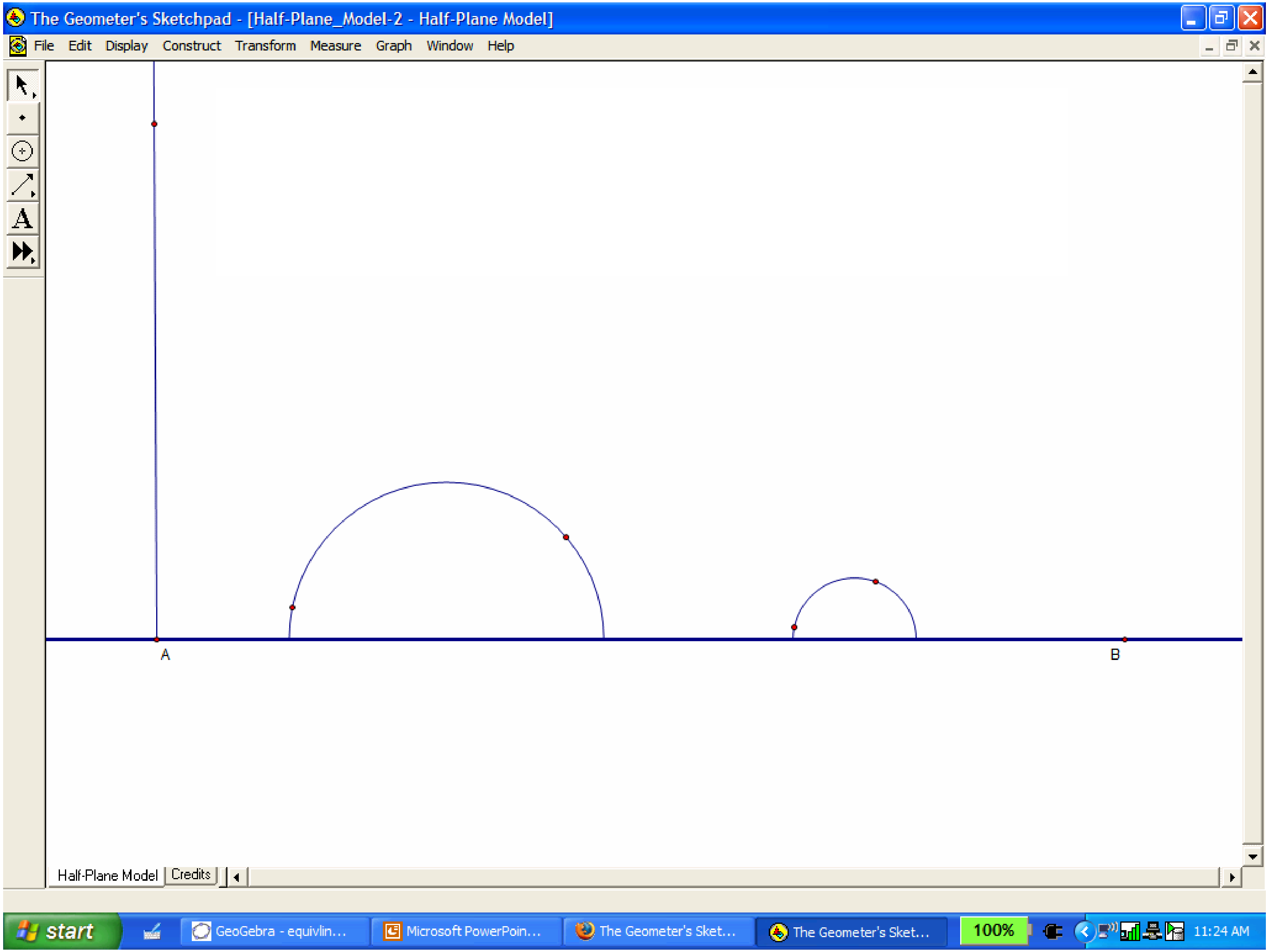
Class #10

Affine and projective planes

Hyperbolic plane (the upper half plane model)

- Points are ordered pairs of real numbers (x, y), where y > 0.

- Lines are
  - Subsets of vertical lines that consist of points (x, y), with y > 0
  - Semicircles whose centers are points (x, 0), where x is a real number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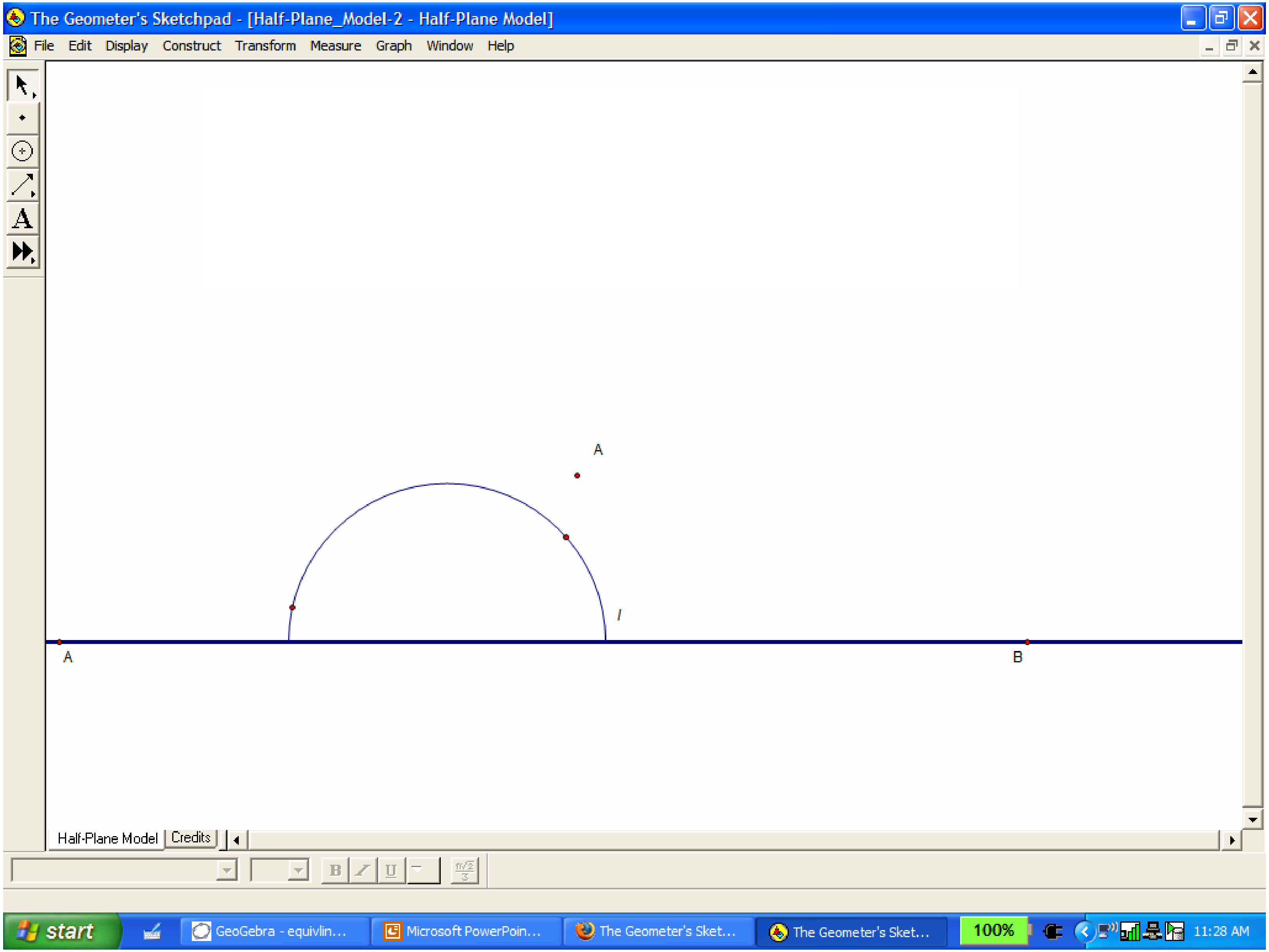
Model #5: H^2

- Hyperbolic plane is also a model of incidence geometry

- It satisfies hyperbolic parallel postulate:
  - For every line *l* and every point P not lying on *l* there are at least two lines that pass through P and are parallel to *l*.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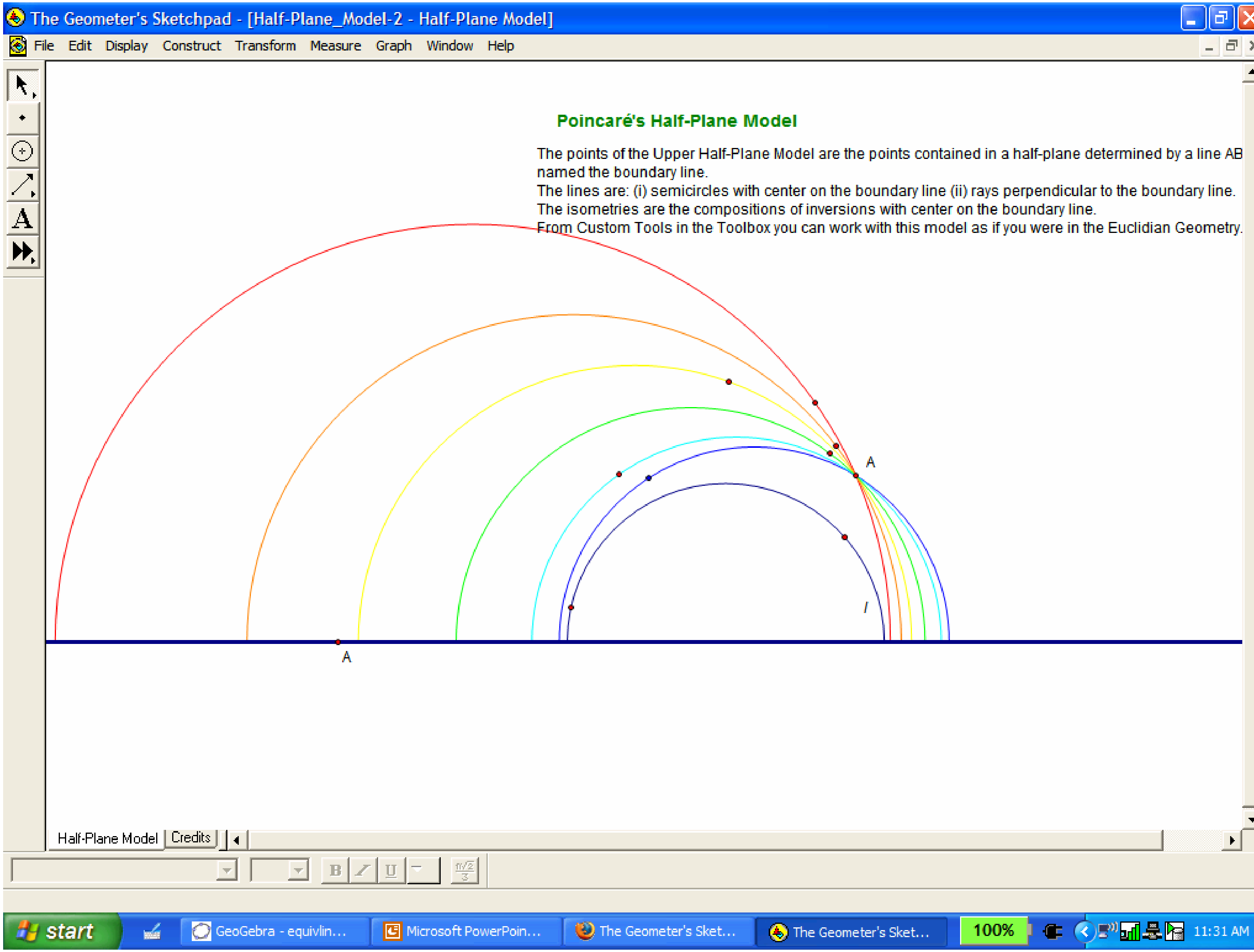
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Poincaré's Half-Plane Model
The points of the Upper Half-Plane Model are the points contained in a half-plane determined by a line AB named the boundary line.
The lines are: (i) semicircles with center on the boundary line (ii) rays perpendicular to the boundary line.
The isometries are the compositions of inversions with center on the boundary line.
From Custom Tools in the Toolbox you can work with this model as if you were in the Euclidian Geometry.
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Affine plane geometry

- The axioms are:
  - ***I-1, I-2, I-3*** & ***EuclideanPP***
- An affine plane is a model of affine plane geometry
- Q: Give two examples of affine planes.
- A:
  - Cartesian plane
  - Model #2: 4 points and 6 lines

Exercise

- Can you prove:
  - **There are four points**.

in incidence geometry?

  - No, because there is a model#1 (in which there are only three points) of incidence geometry in which this statement is clearly incorrect.
- Can you prove it in affine geometry?
  - Proof: By axiom I-3 there exist three distinct points P, Q and R. By axiom I-1 there is a unique line *l* passing through P and Q. By our choice of points P, Q, and R the point R does not lie on *l* (I-3 says that no line is incident with all three points P, Q and R). Euclidean parallel postulate there is a unique line *m* passing through R parallel to *l*. By I-2 there are at least two distinct points on *m*, hence there must exist a point S on *m* different from R. By definition of parallel lines S can not equal P or Q, hence we have found four distinct points: P, Q, R, and S.

Questions to ask when adding an axiom

- Why?
- Is the axiom independent of others?
- Is the new system consistent ?

Consistency

- A system is *consistent* if it is impossible to derive a contradiction.
- Q: Why would being able to derive a contradiction be bad?
- A: Everything follows from contradiction. Every statement you could possibly imagine would be a theorem in that system.

Modified Model#2

- For each set of parallel lines add a new point to the model#2 that will lie on each of those parallel lines. If a line does not have any parallels then add to the model#2 a new point that will lie on that line only.
- Write out all the points and all the lines.
  - Points: A, B, C, D, E, F, G
  - Lines: {A,B,E}, {C,D,E}, {A,C,F}, {B,D,F}, {A,D,G},{B,C,G}
- Is this a model of incidence geometry?
  - No, because the first axiom is not satisfied for points E and F, for example. We need to add another line: {E, F, G}.
- Which parallel postulate holds in this new model?