Brownian motion and thermal capacity[*]

Davar Khoshnevisan
University of Utah

Yimin Xiao
Michigan State University

April 15, 2011

Abstract

Let W denote d-dimensional Brownian motion. We find an explicit formula for the essential supremum of Hausdorff dimension of $W(E) \cap F$, where $E \subset (0\,,\infty)$ and $F \subset \mathbf{R}^d$ are arbitrary nonrandom compact sets. Our formula is related intimately to the thermal capacity of Watson (1978). We prove also that when $d \geq 2$, our formula can be described in terms of the Hausdorff dimension of $E \times F$, where $E \times F$ is viewed as a subspace of space time.

Keywords: Brownian motion, thermal capacity, Euclidean and space-time Hausdorff dimension.

AMS 2000 subject classification: Primary 60J65, 60G17; Secondary 28A78, 28A80, 60G15, 60J45.

1 Introduction

Let $W := \{W(t)\}_{t \geq 0}$ denote a standard d-dimensional Brownian motion, where $d \geq 1$. The principle aim of this paper is to describe the Hausdorff dimension $\dim_{\mathrm{H}}(W(E) \cap F)$ of the random intersection set $W(E) \cap F$, where E and F are compact subsets of $(0\,,\infty)$ and $\mathbf{R}^d$, respectively. This endeavor solves what appears to be an old problem in the folklore of Brownian motion.

In general, the Hausdorff dimension of $W(E) \cap F$ is a random variable, and hence we seek only to compute the $L^\infty(\mathrm{P})$-norm of that Hausdorff dimension. The following example—due to Gregory Lawler—highlights the preceding assertion: Consider $d = 1$, and set $E := \{1\} \cup [2\,,3]$ and $F := [1\,,2]$. Also consider the two events

$$
\begin{aligned}
A_1 &:= \{1 \leq W(1) \leq 2 \,, \; W([2\,,3]) \cap [1\,,2] = \varnothing\}, \\
A_2 &:= \{W(1) \notin [1\,,2] \,, \; W([2\,,3]) \subset [1\,,2]\}.
\end{aligned}
\tag{1.1}
$$

Evidently A_1 and A_2 are disjoint; and each has positive probability. However, $\dim_{\mathrm{H}}(W(E) \cap F) = 0$ on A_1, whereas $\dim_{\mathrm{H}}(W(E) \cap F) = 1$ on A_2. Therefore, $\dim_{\mathrm{H}}(W(E) \cap F)$ is nonconstant, as asserted.

Our first result describes our contribution in the case that $d \geq 2$. In order to describe that contribution let us define ϱ to be the *parabolic metric* on "space time" $\mathbf{R}_+ \times \mathbf{R}^d$; that is,

$$\varrho\left((s\,,x)\,;(t\,,y)\right) := \max\left(|t - s|^{1/2},\ \|x - y\|\right). \tag{1.2}$$

The metric space $\mathbf{S} := (\mathbf{R}_+ \times \mathbf{R}^d, \varrho)$ is also called *space time*, and Hausdorff dimension of the compact set $E \times F$—viewed as a set in $\mathbf{S}$—is denoted by $\dim_{\mathrm{H}}(E \times F\,;\varrho)$. That is, $\dim_{\mathrm{H}}(E \times F\,;\varrho)$ is the largest value of $s \geq 0$ for which $\lim_{\varepsilon \to 0} \inf \sum_{j=1}^{\infty} |\varrho\text{-diam}(E_j \times F_j)|^s$ is infinite, where the infimum is taken over all closed covers $\{E_j \times F_j\}_{j=1}^{\infty}$ of $E \times F$ with $\varrho\text{-diam}(E_j \times F_j) < \varepsilon$, and "$\varrho$-diam($\Lambda$)" denotes the diameter of the space-time set Λ, as measured by the metric ϱ.

Theorem 1.1. *If $d \geq 2$, then*

$$\|\dim_{\mathrm{H}}(W(E) \cap F)\|_{L^{\infty}(\mathrm{P})} = \dim_{\mathrm{H}}(E \times F\,;\varrho) - d, \tag{1.3}$$

where "$\dim_{\mathrm{H}} A < 0$" means "$A = \varnothing$." Display (1.3) continues to hold for $d = 1$, provided that "$=$" is replaced by "$\leq$."

The following example shows that (1.3) does not always hold for $d = 1$: Consider $E := [0\,,1]$ and $F := \{0\}$. Then, a computation on the side shows that $\dim_{\mathrm{H}}(W(E) \cap F) = 0$ a.s., whereas $\dim_{\mathrm{H}}(E \times F\,;\varrho) - d = 1$.

The following result gives a suitable [though quite complicated] formula that is valid for all dimensions, including $d = 1$.

Theorem 1.2. *In general, the following is valid:*

$$\|\dim_{\mathrm{H}}(W(E) \cap F)\|_{L^{\infty}(\mathrm{P})} = \sup\left\{\gamma > 0 : \inf_{\mu \in \mathcal{P}(E \times F)} \mathcal{E}_{\gamma}(\mu) < \infty\right\}; \tag{1.4}$$

where $\mathcal{P}(E \times F)$ denotes the collection of all compactly-supported probability measures on $E \times F$, and

$$\mathcal{E}_{\gamma}(\mu) := \iint \frac{\mathrm{e}^{-\|x-y\|^2/(2|t-s|)}}{|t - s|^{d/2} \cdot \|y - x\|^{\gamma}}\,\mu(\mathrm{d}s\,\mathrm{d}x)\,\mu(\mathrm{d}t\,\mathrm{d}y). \tag{1.5}$$

Theorems 1.1 and 1.2 are the main results of this paper. But it seems natural that we say a few words about when $W(E) \cap F$ is nonvoid with positive probability, simply because when $\mathrm{P}\{W(E) \cap F = \varnothing\} = 1$, there is no point in computing the Hausdorff dimension of $W(E) \cap F$!

According to a theorem of Doob [3], $W(E)$ intersects F with positive probability if and only if $E \times F$ has positive thermal capacity in the sense of Watson

[19, 20]. Doob's result combined with Theorem 3 of Taylor and Watson [18] tells us the following: If

$$\dim_{\mathrm{H}}(E \times F \,; \varrho) > d, \tag{1.6}$$

then $W(E) \cap F$ is nonvoid with positive probability; but if $\dim_{\mathrm{H}}(E \times F \,; \varrho) < d$ then $W(E) \cap F = \varnothing$ almost surely. Kaufman and Wu [9] contain related results. And our Theorem 1.1 states that [the essential sup of] the Hausdorff dimension of $W(E) \cap F$ is the slack in the Taylor–Watson condition (1.6) for the nontriviality of $W(E) \cap F$.

The following yields another interpretation of the assertion that $E \times F$ has positive thermal capacity, and relates one of the energy forms that appear in Theorem 1.2, namely $\mathcal{E}_0$, to the present context.

Proposition 1.3. $\mathrm{P}\{W(E) \cap F \neq \varnothing\} > 0$ *if and only if there exists a probability measure μ on $E \times F$ such that $\mathcal{E}_0(\mu) < \infty$.*

Theorems 1.1 and 1.2 both proceed by checking to see whether or not $W(E) \cap F$ [and a close variant of it] intersect a sufficiently-thin random set. This socalled "codimension idea" has been used with great success in other situations as well [5, 12, 15, 17]. To the best of our knowledge, the broad utility of this method—using fractal percolation sets as the [thin] testing random sets—was pointed out first by Yuval Peres [14].

Throughout this paper we adopt the following notation: For all integers $k \geq 1$ and for every $x = (x_1, \ldots, x_k) \in \mathbf{R}^k$, $\|x\|$ and $|x|$ respectively define the ℓ^2 and ℓ^1 norms of x. That is,

$$\|x\| := \left(x_1^2 + \cdots + x_k^2\right)^{1/2} \quad \text{and} \quad |x| := |x_1| + \cdots + |x_k|. \tag{1.7}$$

After a brief derivation of Proposition 1.3, Theorems 1.1 and 1.2 are proved in reverse order, since the latter is significantly harder to prove.

2 Proof of Proposition 1.3

It is possible to prove this proposition from first principles, but we only know of one such proof; and it is quite long. Instead, we opt for a short argument that rests on quite-deep facts about the general theory of Markov processes.

We can assume, without loss of too much generality, that $E \subset [1 \,, 2]$. It is not difficult to adapt the ensuing proof to the general setting where E is an arbitrary compact subset of $(0 \,, \infty)$.

Consider the Doob process $D := \{D(t)\}_{t \in [0,2]}$, where

$$D(t) := (t \,, W(t)). \tag{2.1}$$

The process D is a [degenerate] two-dimensional diffusion indexed by the time interval $[0 \,, 2]$; it is also a quite nice Hunt process. In addition, we might note that the probability that we want is the hitting probability of $E \times F$ by D; that is,

$$\mathrm{P}\left\{W(E) \cap F \neq \varnothing\right\} = \mathrm{P}\left\{D(t) \in E \times F \text{ for some } t \in [0 \,, 2]\right\}. \tag{2.2}$$

3

Because of translation invariance, it is easy to see that the 0-potential of D is described by

$$u(t,x) := \frac{\mathrm{e}^{-\|x\|^2/(2t)}}{(2\pi t)^{d/2}} \mathbf{1}_{(0,\infty)}(t), \qquad (2.3)$$

for every $(t,x) \in [0,2] \times \mathbf{R}^d$. That is, for all bounded Borel-measurable functions $f : [0,2] \times \mathbf{R}^d \to \mathbf{R}_+$,

$$\mathrm{E} \int_0^2 f(D(t))\,\mathrm{d}t = \int_{[0,2] \times \mathbf{R}^d} f(t,x)u(t,x)\,\mathrm{d}t\,\mathrm{d}x. \qquad (2.4)$$

Therefore, the 0-potential $U\mu$ of a Borel measure μ on $\mathbf{R}_+ \times \mathbf{R}^d$ is[1]

$$\begin{aligned}
(U\mu)(s,y) &:= \int u(t-s,y-x)\,\mu(\mathrm{d}t\,\mathrm{d}x) \\
&= \int_{t>s} \frac{\mathrm{e}^{-\|x-y\|^2/(2(t-s))}}{(2(t-s)\pi)^{d/2}}\,\mu(\mathrm{d}t\,\mathrm{d}x).
\end{aligned} \qquad (2.5)$$

According to Watson [19, Lemma 4], $E \times F$ has positive thermal capacity if and only if there exists a probability measure μ on $E \times F$ such that $U\mu$ is a bounded function on $\mathbf{R}_+ \times \mathbf{R}^d$. When combined with Doob's theorem [3], the preceding leads us to the following dichotomy:

$$\mathrm{P}\{W(E) \cap F \neq \varnothing\} > 0 \quad \Leftrightarrow \quad \inf_{\mu \in \mathcal{P}(E \times F)} \sup_{[0,2] \times \mathbf{R}^d} U\mu < \infty. \qquad (2.6)$$

The natural dual process to D is $\hat{D} := \{\hat{D}(t)\}_{t \in [0,2]}$, where

$$\hat{D}(t) := (2 - t, W(2) - W(2 - t)). \qquad (2.7)$$

Since D and $\hat{D}$ have the same finite-dimensional distributions, they are "locally symmetric" in the sense of Orey [16]. Therefore, Exercise **(1.26)** of Blumenthal and Getoor [2, p. 265] tells us that for every compact set $K \subset [0,2] \times \mathbf{R}^d$ we have the "maximum principle" on K; that is,

$$\sup_{[0,2] \times \mathbf{R}^d} U\mu = \sup_K U\mu, \qquad (2.8)$$

valid for every finite Borel measure μ on K. For more details, see the subsequent paper [1] by Blumenthal and Getoor.

If $\mu \in \mathcal{P}(K)$ for some compact set $K \subset [0,2] \times \mathbf{R}^d$, then $\frac{1}{2}\mathcal{E}_0(\mu) \leq \int (U\mu)\,\mathrm{d}\mu \leq \mathcal{E}_0(\mu)$ and $\int (U\mu)\,\mathrm{d}\mu \leq \sup_K U\mu$. Therefore, it suffices to prove that

$$\inf_{\nu \in \mathcal{P}(E \times F)} \int (U\nu)\,\mathrm{d}\nu < \infty \quad \Rightarrow \quad \inf_{\nu \in \mathcal{P}(E \times F)} \sup_{[0,2] \times \mathbf{R}^d} U\nu < \infty, \qquad (2.9)$$

---

[1]These are the thermal potentials of Watson [20]. But Watson uses Δ in place of $\frac{1}{2}\Delta$, and hence obtains $4(t-s)$ in place of $2(t-s)$ in (2.5).

the other direction being trivial. One can obtain this from the general theory of Markov processes as well, but here is a more efficient argument: If $\int (U\nu)\,d\nu < \infty$ for some $\nu \in \mathcal{P}(E \times F)$, then thanks to Lusin's theorem we can find a compact set K in $E \times F$ such that: (i) $\nu(K) > 0$; and (ii) $U\nu$ is uniformly continuous on K. Let ν' denote the restriction of ν to K, normalized to have mass one. According to the maximum principle,

$$\sup_{[0,2] \times \mathbf{R}^d} U\nu' = \sup_K U\nu' \leq \frac{1}{\nu(K)} \sup_K U\nu < \infty. \tag{2.10}$$

This completes the proof because ν' is a probability measure on $E \times F$. $\qquad \square$

3 Proof of Theorem 1.2

Here and throughout,

$$B_x(\epsilon) := \left\{ y \in \mathbf{R}^d : \|x - y\| \leq \epsilon \right\} \tag{3.1}$$

denotes the radius-ϵ ball about $x \in \mathbf{R}^d$. Also, define ν_d to be the volume of $B_0(1)$; that is,

$$\nu_d := \frac{2 \cdot \pi^{d/2}}{d\Gamma(d/2)}. \tag{3.2}$$

Recall that $t \mapsto W(t)$ defines a Brownian motion in $\mathbf{R}^d$, and consider the following for all $\epsilon, t > 0$ and $x \in \mathbf{R}^d$:

$$p_t(x) := \frac{\mathrm{e}^{-\|x\|^2/(2t)}}{(2\pi t)^{d/2}} \mathbf{1}_{(0,\infty)}(t). \tag{3.3}$$

[The seemingly-innocuous indicator function plays an important role in the sequel; this form of the heat kernel appears earlier in the original papers by Watson [20, 19].] Recall that $\mathcal{P}(U)$ denotes the collection of all Borel probability measures that are supported on a compact subset of U. Then whenever $\mu \in \mathcal{P}(\mathbf{R}_+ \times \mathbf{R}^d)$, its *thermal* [or *heat*] *energy* is defined by $\mathcal{E}_0(\mu)$. The *thermal capacity* of the set $E \times F$ is then defined by

$$\mathcal{C}_0(E \times F) := \left[\inf_{\mu \in \mathcal{P}(E \times F)} \mathcal{E}_0(\mu) \right]^{-1}, \tag{3.4}$$

where $\inf \varnothing := \infty$ and $1/\infty := 0$.

Independently of W, we introduce N isotropic stable processes $\{X^{(j)}\}_{j=1}^N$, each with index $\alpha \in (0, 2]$. We assume that the $X^{(j)}$'s are totally independent from one another, as well as the process W, and all take their values in $\mathbf{R}^d$. We assume also that $X^{(1)}, \ldots, X^{(N)}$ have right-continuous sample paths with left-limits. This assumption can be—and will be—made without incurring any real loss in generality. Finally, our normalization of the processes $X^{(1)}, \ldots, X^{(N)}$ is described as follows:

$$\mathrm{E}\left[\exp\left\{ i\xi \cdot X^{(\ell)}(1) \right\} \right] = \mathrm{e}^{-\|\xi\|^\alpha/2} \quad \text{for all } 1 \leq \ell \leq N \text{ and } \xi \in \mathbf{R}^d. \tag{3.5}$$

Define the corresponding *additive stable process* $X_\alpha := \{X_\alpha(t)\}_{t\in\mathbf{R}_+^N}$ as

$$X_\alpha(t) := \sum_{k=1}^{N} X^{(k)}(t_k) \qquad \text{for all} \qquad t := (t_1, \ldots, t_N) \in \mathbf{R}_+^N. \qquad (3.6)$$

Also, define $\mathcal{C}_\gamma$ to be the capacity corresponding to the energy form (1.5). That is, for all compact sets $U \subset \mathbf{R}_+ \times \mathbf{R}^d$ and $\gamma \geq 0$,

$$\mathcal{C}_\gamma(U) := \left[\inf_{\mu\in\mathcal{P}(U)} \mathcal{E}_\gamma(\mu)\right]^{-1}. \qquad (3.7)$$

Theorem 3.1. *If $d > \alpha N$, then*

$$\mathrm{P}\left\{W(E) \cap \overline{X_\alpha(\mathbf{R}_+^N)} \cap F \neq \varnothing\right\} > 0 \quad \Longleftrightarrow \quad \mathcal{C}_{d-\alpha N}(E \times F) > 0. \qquad (3.8)$$

Here and in the sequel, $\overline{A}$ denotes the closure of A.

Remark 3.2. It can be proved that the same result continues to hold even if we remove the closure sign. We will not delve into this here because we do not need the said refinement. $\square$

We can now apply Theorem 3.1 to prove Theorem 1.2. Theorem 3.1 will be established subsequently.

Proof of Theorem 1.2. Suppose $\alpha \in (0, 2]$ and $N \in \mathbf{Z}_+$ are chosen such that $d > \alpha N$. If X_α denotes an N-parameter additive stable process $\mathbf{R}^d$ whose index is $\alpha \in (0, 2]$, then [10, Example 2, p. 436]

$$\mathrm{codim}\, \overline{X_\alpha(\mathbf{R}_+^N)} = d - \alpha N. \qquad (3.9)$$

This means that the closure of $X_\alpha(\mathbf{R}_+^N)$ will intersect any nonrandom Borel set $G \subset \mathbf{R}^d$ with $\dim_{\mathrm{H}}(G) > d - \alpha N$, whereas any G with $\dim_{\mathrm{H}}(G) < d - \alpha N$ does not intersect the closure of $X_\alpha(\mathbf{R}_+^N)$.

Define

$$\Delta := \sup\left\{\gamma > 0 : \inf_{\mu\in\mathcal{P}(E\times F)} \mathcal{E}_\gamma(\mu) < \infty\right\}. \qquad (3.10)$$

If $d - \alpha N < \Delta$, then $\mathcal{C}_{d-\alpha N}(E \times F) > 0$. Therefore, it follows from Theorem 3.1 and (3.9) that if $\mathrm{P}\{W(E) \cap F \neq \varnothing\} > 0$, then

$$d - \alpha N < \Delta \;\Rightarrow\; \mathrm{P}\left\{\dim_{\mathrm{H}}(W(E) \cap F) \geq d - \alpha N\right\} > 0. \qquad (3.11)$$

Similarly,

$$d - \alpha N > \Delta \;\Rightarrow\; \dim_{\mathrm{H}}(W(E) \cap F) \leq d - \alpha N \quad \text{almost surely.} \qquad (3.12)$$

Because $d - \alpha N \in (0, d)$ is arbitrary, (3.11) implies that

$$\|\dim_{\mathrm{H}}(W(E) \cap F)\|_{L^\infty(\mathrm{P})} \geq \Delta \quad \text{when} \quad \mathrm{P}\{W(E) \cap F \neq \varnothing\} > 0. \qquad (3.13)$$

This is half of the theorem.

Similarly, if $\Delta > 0$, then (3.12) implies that $\|\dim_{\mathrm{H}}(W(E) \cap F)\|_{L^\infty(\mathrm{P})} \leq \Delta$. Thus, it remains to investigate the case $\Delta = 0$. But in this case, we see that no matter how we choose $\alpha \in (0, 2]$ and $N \in \mathbf{Z}_+$ to make $d - \alpha N > 0$, we necessarily have

$$\overline{X_\alpha(\mathbf{R}_+^N)} \cap W(E) \cap F = \varnothing \qquad \text{almost surely.} \qquad (3.14)$$

Therefore, it follows from (3.9) that $\dim_{\mathrm{H}}(W(E) \cap F) \leq d - \alpha N$ almost surely. Because we can choose $d - \alpha N$ to be as small as we wish, this proves the theorem. $\qquad\square$

4 Proof of Theorem 3.1

Our proof of Theorem 3.1 is divided into separate parts. We begin by developing a requisite result in harmonic analysis. Then, we develop some facts about additive Lévy processes. After that, we prove Theorem 3.1 in two separate parts.

4.1 Isoperimetry

Recall that a function $\kappa : \mathbf{R}^n \to \bar{\mathbf{R}}_+$ is *tempered* if is measurable and

$$\int_{\mathbf{R}^n} \frac{\kappa(x)}{(1 + \|x\|)^m} \, \mathrm{d}x < \infty \qquad \text{for some } m \geq 0. \qquad (4.1)$$

A function $\kappa : \mathbf{R}^n \to \bar{\mathbf{R}}_+$ is said to be *positive definite* if it is tempered and for all rapidly-decreasing test functions $\phi : \mathbf{R}^n \to \mathbf{R}$,

$$\int_{\mathbf{R}^n} \mathrm{d}x \int_{\mathbf{R}^n} \mathrm{d}y \ \phi(x)\kappa(x - y)\phi(y) \geq 0. \qquad (4.2)$$

We make heavy use of the following result of Foondun and Khoshnevisan [4, Corollary 3.7]; for a weaker version see [11, Theorem 5.2].

Lemma 4.1. *If $\kappa : \mathbf{R}^n \to \bar{\mathbf{R}}_+$ is positive definite and lower semicontinuous, then for all finite Borel measures μ on $\mathbf{R}^n$,*

$$\iint \kappa(x - y) \, \mu(\mathrm{d}x) \, \mu(\mathrm{d}y) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \hat{\kappa}(\xi)|\hat{\mu}(\xi)|^2 \, \mathrm{d}\xi. \qquad (4.3)$$

Lemma 4.1 implies two "isoperimetric inequalities," that are stated below as Propositions 4.2 and 4.4. Recall that a finite Borel measure ν on $\mathbf{R}^d$ is said to be *positive definite* if $\hat{\nu}(\xi) \geq 0$ for all $\xi \in \mathbf{R}^d$.

Proposition 4.2. *Suppose $\kappa : \mathbf{R}^d \to \bar{\mathbf{R}}_+$ is a lower semicontinuous positive-definite function such that $\kappa(0) = \infty$. Suppose ν and σ are two positive definite probability measures on $\mathbf{R}^d$ that satisfy the following:*

1. κ and $\kappa * \nu$ are uniformly continuous on every compact subset of $\{x \in \mathbf{R}^d : \|x\| \geq \eta\}$ for each $\eta > 0$; and

2. $(\tau, x) \mapsto (p_\tau * \sigma)(x)$ is uniformly continuous on every compact subset of $\{(t, x) \in \mathbf{R}_+ \times \mathbf{R}^d : t \wedge \|x\| \geq \eta\}$ for each $\eta > 0$.

Then, for all finite Borel measures μ on $\mathbf{R}_+ \times \mathbf{R}^d$,

$$
\iint (p_{|t-s|} * \sigma)(x - y)(\kappa * \nu)(x - y) \, \mu(\mathrm{d}t \, \mathrm{d}x) \, \mu(\mathrm{d}s \, \mathrm{d}y)
$$
$$
\leq \iint p_{|t-s|}(x - y)\kappa(x - y) \, \mu(\mathrm{d}t \, \mathrm{d}x) \, \mu(\mathrm{d}s \, \mathrm{d}y).
\tag{4.4}
$$

Remark 4.3. The very same proof shows the following slight enhancement: *Suppose the conditions of Proposition 4.2 are met. If σ_1 and σ_2 share the properties of σ in Proposition 4.2 and $\hat{\sigma}_1(\xi) \leq \hat{\sigma}_2(\xi)$ for all $\xi \in \mathbf{R}^d$, then for all finite Borel measures μ on $\mathbf{R}_+ \times \mathbf{R}^d$,*

$$
\iint (p_{|t-s|} * \sigma_1)(x - y)(\kappa * \nu)(x - y) \, \mu(\mathrm{d}t \, \mathrm{d}x) \, \mu(\mathrm{d}s \, \mathrm{d}y)
$$
$$
\leq \iint (p_{|t-s|} * \sigma_2)(x - y)\kappa(x - y) \, \mu(\mathrm{d}t \, \mathrm{d}x) \, \mu(\mathrm{d}s \, \mathrm{d}y).
\tag{4.5}
$$

Proposition 4.2 is this in the case that $\sigma_2 := \delta_0$. Analogous result holds for positive definite probability measures ν_1 and ν_2 which satisfy $\hat{\nu}_1(\xi) \leq \hat{\nu}_2(\xi)$ for all $\xi \in \mathbf{R}^d$. $\qquad \square$

Proof. Throughout this proof, we choose and fix $\epsilon > 0$.

Without loss of generality, we may and will assume that

$$
\iint p_{|t-s|}(x - y)\kappa(x - y) \, \mu(\mathrm{d}t \, \mathrm{d}x) \, \mu(\mathrm{d}s \, \mathrm{d}y) < \infty;
\tag{4.6}
$$

for there is nothing to prove, otherwise.

Because $p_{|t-s|}$ is positive definite for every nonnegative $t \neq s$, so are $p_{|t-s|} * \sigma$ and $\kappa * \nu$. Products of positive-definite functions are positive definite themselves. Therefore, for fixed $t > s$, Lemma 4.1 applies, and tells us that for all Borel probability measures ρ on $\mathbf{R}^d$, and for all nonnegative $t \neq s$,

$$
\iint (p_{|t-s|} * \sigma)(x - y)(\kappa * \nu)(x - y) \, \rho(\mathrm{d}x) \, \rho(\mathrm{d}y)
$$
$$
= \frac{1}{(2\pi)^{2d}} \int_{\mathbf{R}^d} \mathrm{d}\xi \int_{\mathbf{R}^d} \mathrm{d}\zeta \; \mathrm{e}^{-(t-s)\|\xi\|^2/2} \hat{\sigma}(\xi)\hat{\kappa}(\zeta)\hat{\nu}(\xi)|\hat{\rho}(\xi - \zeta)|^2.
\tag{4.7}
$$

Because the preceding is valid also when $\sigma = \nu = \delta_0$, and since $0 \leq \hat{\sigma}(\xi), \hat{\nu}(\xi) \leq 1$ for all $\xi \in \mathbf{R}^d$, it follows that for all nonnegative $t \neq s$,

$$
\iint (p_{|t-s|} * \sigma)(x - y)(\kappa * \nu)(x - y) \, \rho(\mathrm{d}x) \, \rho(\mathrm{d}y)
$$
$$
\leq \iint p_{|t-s|}(x - y)\kappa(x - y) \, \rho(\mathrm{d}x) \, \rho(\mathrm{d}y).
\tag{4.8}
$$

This inequality continues to holds when ρ is a finite Borel measure on $\mathbf{R}^d$, by scaling. Thus, thanks to Tonelli's theorem, the proposition is valid whenever $\mu(\mathrm{d}t\,\mathrm{d}x) = \lambda(\mathrm{d}t)\rho(\mathrm{d}x)$ for two finite Borel measures λ and ρ, respectively defined on $\mathbf{R}_+$ and $\mathbf{R}^d$.

Now let us consider an compactly-supported finite measure μ on $\mathbf{R}_+ \times \mathbf{R}^d$. For all $\eta > 0$ define

$$\mathcal{G}(\eta) := \left\{ (t\,,s\,,x\,,y) \in (\mathbf{R}_+)^2 \times (\mathbf{R}^d)^2 : \ |t - s| \wedge \|x - y\| \geq \eta \right\}. \tag{4.9}$$

It suffices to prove that for all $\eta > 0$,

$$\iint_{\mathcal{G}(\eta)} (p_{|t-s|} * \sigma)(x - y)(\kappa * \nu)(x - y)\,\mu(\mathrm{d}t\,\mathrm{d}x)\,\mu(\mathrm{d}s\,\mathrm{d}y)$$
$$\leq \iint_{\mathcal{G}(\eta)} p_{|t-s|}(x - y)\kappa(x - y)\,\mu(\mathrm{d}t\,\mathrm{d}x)\,\mu(\mathrm{d}s\,\mathrm{d}y). \tag{4.10}$$

This is so, because $\kappa(0) = \infty$ and (4.6) readily tell us that $\mu \times \mu$ does not charge

$$\left\{ (t\,,s\,,x\,,y) \in (\mathbf{R}_+)^2 \times (\mathbf{R}^d)^2 : \ x = y \right\}; \tag{4.11}$$

and therefore,

$$\lim_{\eta \downarrow 0} \iint_{\mathcal{G}(\eta)} (p_{|t-s|} * \sigma)(x - y)(\kappa * \nu)(x - y)\,\mu(\mathrm{d}t\,\mathrm{d}x)\,\mu(\mathrm{d}s\,\mathrm{d}y)$$
$$= \iint_{\substack{s \neq t \\ x \neq y}} (p_{|t-s|} * \sigma)(x - y)(\kappa * \nu)(x - y)\,\mu(\mathrm{d}t\,\mathrm{d}x)\,\mu(\mathrm{d}s\,\mathrm{d}y) \tag{4.12}$$
$$= \iint (p_{|t-s|} * \sigma)(x - y)(\kappa * \nu)(x - y)\,\mu(\mathrm{d}t\,\mathrm{d}x)\,\mu(\mathrm{d}s\,\mathrm{d}y).$$

And similarly,

$$\lim_{\eta \downarrow 0} \iint_{\mathcal{G}(\eta)} p_{|t-s|}(x - y)\kappa(x - y)\,\mu(\mathrm{d}t\,\mathrm{d}x)\,\mu(\mathrm{d}s\,\mathrm{d}y)$$
$$= \iint p_{|t-s|}(x - y)\kappa(x - y)\,\mu(\mathrm{d}t\,\mathrm{d}x)\,\mu(\mathrm{d}s\,\mathrm{d}y). \tag{4.13}$$

And the proposition follows.

Next we verify (4.10) to finish the proof.

One can check directly that $\mathcal{G}(\eta) \cap \mathrm{supp}(\mu \times \mu)$ is compact, and both mappings $(t\,,s\,,x\,,y) \mapsto (p_{|t-s|} * \sigma)(x - y)(\kappa * \nu)(x - y)$ and $(t\,,s\,,x\,,y) \mapsto p_{|t-s|}(x - y)\kappa(x - y)$ are uniformly continuous on $\mathcal{G}(\eta) \cap \mathrm{supp}(\mu \times \mu)$.

We can find finite Borel measures $\{\lambda_j\}_{j=1}^{\infty}$—on $\mathbf{R}_+$—and $\{\rho_j\}_{j=1}^{\infty}$—on $\mathbf{R}^d$—such that μ is the weak limit of $\mu_N := \sum_{j=1}^{N} (\lambda_j \times \rho_j)$ as $N \to \infty$. The already-proved portion of this proposition implies that for all $\eta > 0$ and $N \geq 1$,

$$\iint_{\mathcal{G}(\eta)} (p_{|t-s|} * \sigma)(x - y)(\kappa * \nu)(x - y)\,\mu_N(\mathrm{d}t\,\mathrm{d}x)\,\mu_N(\mathrm{d}s\,\mathrm{d}y)$$
$$\leq \iint_{\mathcal{G}(\eta)} p_{|t-s|}(x - y)\kappa(x - y)\,\mu_N(\mathrm{d}t\,\mathrm{d}x)\,\mu_N(\mathrm{d}s\,\mathrm{d}y). \tag{4.14}$$

9

Let $N \uparrow \infty$ to deduce (4.10) and hence the proposition. $\qquad \square$

Proposition 4.4. *Suppose $\kappa : \mathbf{R} \to \bar{\mathbf{R}}_+$ is a lower semicontinuous positive-definite function. Suppose ν and σ are two positive definite probability measures, respectively on $\mathbf{R}$ and $\mathbf{R}^d$, that satisfy the following:*

1. *κ and $\kappa * \nu$ are uniformly continuous on every compact subset of $\{x \in \mathbf{R} : \|x\| \geq \eta\}$ for each $\eta > 0$; and*

2. *$(\tau, x) \mapsto (p_\tau * \sigma)(x)$ is uniformly continuous on every compact subset of $\{(t, x) \in \mathbf{R}_+ \times \mathbf{R}^d : t, \|x\| \geq \eta\}$ for each $\eta > 0$.*

Then, for all finite Borel measures μ on $\mathbf{R}_+ \times \mathbf{R}^d$,

$$
\iint (p_{|t-s|} * \sigma)(x - y)(\kappa * \nu)(s - t) \, \mu(\mathrm{d}t \, \mathrm{d}x) \, \mu(\mathrm{d}s \, \mathrm{d}y)
$$
$$
\leq \iint p_{|t-s|}(x - y)\kappa(s - t) \, \mu(\mathrm{d}t \, \mathrm{d}x) \, \mu(\mathrm{d}s \, \mathrm{d}y). \tag{4.15}
$$

Proof. It suffices to prove the proposition in the case that

$$
\mu(\mathrm{d}s \, \mathrm{d}x) = \lambda(\mathrm{d}s)\rho(\mathrm{d}x), \tag{4.16}
$$

for finite Borel measures λ and ρ, respectively on $\mathbf{R}_+$ and $\mathbf{R}^d$. See, for instance, the argument beginning with (4.9) in the proof of Proposition 4.2. We shall extend the definition λ so that it is a finite Borel measure on all of $\mathbf{R}$ in the usual way: If $A \subset \mathbf{R}$ is Borel measurable, then $\lambda(A) := \lambda(A \cap \mathbf{R}_+)$. This slight abuse in notation should not cause any confusion in the sequel.

Tonelli's theorem and Lemma 4.1 together imply that in the case that (4.16) holds,

$$
\iint (p_{|t-s|} * \sigma)(x - y)(\kappa * \nu)(s - t) \, \mu(\mathrm{d}t \, \mathrm{d}x) \, \mu(\mathrm{d}s \, \mathrm{d}y) \tag{4.17}
$$
$$
= \iint \lambda(\mathrm{d}t) \, \lambda(\mathrm{d}s) \, (\kappa * \nu)(s - t) \iint \rho(\mathrm{d}x) \, \rho(\mathrm{d}y) \, (p_{|t-s|} * \sigma)(x - y)
$$
$$
= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \hat{\sigma}(\xi)|\hat{\rho}(\xi)|^2 \, \mathrm{d}\xi \iint \lambda(\mathrm{d}t) \, \lambda(\mathrm{d}s) \, (\kappa * \nu)(s - t)\mathrm{e}^{-|t-s| \cdot \|\xi\|^2/2}
$$
$$
\leq \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} |\hat{\rho}(\xi)|^2 \, \mathrm{d}\xi \iint \lambda(\mathrm{d}t) \, \lambda(\mathrm{d}s) \, (\kappa * \nu)(s - t)\mathrm{e}^{-|t-s| \cdot \|\xi\|^2/2}.
$$

The map $\tau \mapsto \exp\{-|\tau| \cdot \|\xi\|^2/2\}$ is positive definite on $\mathbf{R}$ for every fixed $\xi \in \mathbf{R}^d$; in fact, its inverse Fourier transform is a [scaled] Cauchy density function, which we refer to as ϑ_ξ. Therefore, in accord with Lemma 4.1,

$$
\iint (\kappa * \nu)(s - t)\mathrm{e}^{-|t-s| \cdot \|\xi\|^2/2} \, \lambda(\mathrm{d}t) \, \lambda(\mathrm{d}s)
$$
$$
= \frac{1}{(2\pi)^d} \int_{\mathbf{R}} |\hat{\lambda}(\tau)|^2(\hat{\kappa}\hat{\nu} * \vartheta_\xi)(\tau) \, \mathrm{d}\tau \leq \frac{1}{(2\pi)^d} \int_{\mathbf{R}} |\hat{\lambda}(\tau)|^2(\hat{\kappa} * \vartheta_\xi)(\tau) \, \mathrm{d}\tau \tag{4.18}
$$
$$
= \iint \kappa(s - t)\mathrm{e}^{-|t-s| \cdot \|\xi\|^2/2} \, \lambda(\mathrm{d}t) \, \lambda(\mathrm{d}s).
$$

10

The last line follows from the first identity, since we can consider $\nu = \delta_0$ as a possibility. Therefore, it follows from (4.17) that

$$\iint (p_{|t-s|} * \sigma)(x - y)(\kappa * \nu)(s - t) \, \mu(\mathrm{d}t \, \mathrm{d}x) \, \mu(\mathrm{d}s \, \mathrm{d}y) \qquad (4.19)$$

$$\leq \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} |\hat{\rho}(\xi)|^2 \, \mathrm{d}\xi \iint \lambda(\mathrm{d}t) \, \lambda(\mathrm{d}s) \, \kappa(s - t) \mathrm{e}^{-|t-s| \cdot \|\xi\|^2/2}$$

$$= \iint \lambda(\mathrm{d}t) \, \lambda(\mathrm{d}s) \, \kappa(s - t) \iint \rho(\mathrm{d}x) \, \rho(\mathrm{d}y) \, p_{|t-s|}(x - y); \qquad (4.20)$$

the last line follows from the first identity in (4.17) by considering the special case that $\nu = \delta_0$. This proves the proposition in the case that μ has the form (4.16), and the result follows. $\qquad \square$

4.2 Additive stable processes

In this subsection we develop a "resolvent density" estimate for the additive stable process X_α.

First of all, note that the characteristic function $\xi \mapsto \mathrm{E} \exp(i\xi \cdot X_\alpha(\boldsymbol{t}))$ of $X_\alpha(\boldsymbol{t})$ is absolutely integrable for every $\boldsymbol{t} \in \mathbf{R}_+^N \setminus \{\boldsymbol{0}\}$. Consequently, the inversion formula applies and tells us that we can always choose the following as the probability density function of $X_\alpha(\boldsymbol{t})$:

$$g_{\boldsymbol{t}}(x) := g_{\boldsymbol{t}}(\alpha\,;x) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \mathrm{e}^{-i(x \cdot \xi) - |\boldsymbol{t}| \cdot \|\xi\|^\alpha/2} \, \mathrm{d}\xi. \qquad (4.21)$$

Lemma 4.5. *Choose and fix some $\boldsymbol{a}, \boldsymbol{b} \in (0\,,\infty)^N$ such that $a_j \leq b_j$ for all $1 \leq j \leq N$. Define*

$$[\boldsymbol{a}\,,\boldsymbol{b}] := \left\{ \boldsymbol{s} \in \mathbf{R}_+^N : a_j \leq s_j \leq b_j \ \ \text{for all} \ \ 1 \leq j \leq N \right\}. \qquad (4.22)$$

Then, for all $M > 0$ there exists a constant $A_0 \in (1\,,\infty)$—depending only on the parameters d, N, M, α, $\min_{1 \leq j \leq N} a_j$, and $\max_{1 \leq j \leq N} b_j$—such that for all $x \in [-M\,,M]^d$,

$$A_0^{-1} \leq \int_{[\boldsymbol{a},\boldsymbol{b}]} g_{\boldsymbol{t}}(x) \, \mathrm{d}\boldsymbol{t} \leq A_0. \qquad (4.23)$$

Proof. Let

$$\mathbf{1} := (1\,,\dots,1) \qquad [N \text{ times}]. \qquad (4.24)$$

Then, we may also observe the scaling relation,

$$g_{\boldsymbol{t}}(x) = |\boldsymbol{t}|^{-d/\alpha} g_{\mathbf{1}}\left(\frac{x}{|\boldsymbol{t}|^{1/\alpha}}\right), \qquad (4.25)$$

together with the fact $g_{\mathbf{1}}$ is an isotropic stable-α density function on $\mathbf{R}^d$. The upper bound in (4.23) follows from (4.25) and the boundedness of $g_{\mathbf{1}}(z)$.

On the other hand, the lower bound in (4.23) follows from (4.25) and the following standard estimate: For all $R > 0$ there exists $C(R) \in (1, \infty)$ and $c(R) \in (0, 1)$ such that

$$\frac{c(R)}{\|z\|^{d+\alpha}} \leq g_{\mathbf{1}}(z) \leq \frac{C(R)}{\|z\|^{d+\alpha}} \qquad \text{for all } z \in \mathbf{R}^d \text{ with } \|z\| \geq R. \tag{4.26}$$

See [10, Proposition 3.3.1, p. 380], where this is proved for $R = 2$. The slightly more general case where $R > 0$ is arbitrary is proved in exactly the same manner. $\qquad\square$

Proposition 4.6. *Choose and fix some $\mathbf{b} \in (0, \infty)^N$ and define $[\mathbf{0}, \mathbf{b}]$ as in Lemma 4.5. Then, for all $M > 0$ there exists a constant $A_1 \in (1, \infty)$—depending only on d, N, M, α, $\min_{1 \leq j \leq N} b_j$, and $\max_{1 \leq j \leq N} b_j$—such that for all $x \in [-M, M]^d$,*

$$\frac{1}{A_1 \|x\|^{d-\alpha N}} \leq \int_{[\mathbf{0}, \mathbf{b}]} g_{\mathbf{t}}(x) \, \mathrm{d}\mathbf{t} \leq \frac{A_1}{\|x\|^{d-\alpha N}}. \tag{4.27}$$

Proof. Since

$$\int_{[\mathbf{0}, \mathbf{b}]} g_{\mathbf{t}}(x) \, \mathrm{d}\mathbf{t} \leq \mathrm{e}^{|\mathbf{b}|} \int_{\mathbf{R}_+^N} \mathrm{e}^{-|\mathbf{t}|} g_{\mathbf{t}}(x) \, \mathrm{d}\mathbf{t}, \tag{4.28}$$

the proof of Proposition 4.1.1 of [10, p. 420] shows that the upper bound in (4.27) holds for all $x \in \mathbf{R}^d$.

For the lower bound, we apply (4.25) and then (4.26) in order to find that

$$\begin{aligned}
\int_{[\mathbf{0}, \mathbf{b}]} g_{\mathbf{t}}(x) \, \mathrm{d}\mathbf{t} &= \int_{[\mathbf{0}, \mathbf{b}]} |\mathbf{t}|^{-d/\alpha} g_{\mathbf{1}} \left(\frac{x}{|\mathbf{t}|^{1/\alpha}} \right) \, \mathrm{d}\mathbf{t} \\
&\geq \frac{c(1)}{\|x\|^{d+\alpha}} \cdot \int_{\substack{\mathbf{t} \in [\mathbf{0}, \mathbf{b}]: \\ |\mathbf{t}|^{1/\alpha} \leq \|x\|}} |\mathbf{t}| \, \mathrm{d}\mathbf{t}.
\end{aligned} \tag{4.29}$$

Clearly, there exists $R_0 > 0$ sufficiently small such that whenever $\|x\| \leq R_0$,

$$\int_{\substack{\mathbf{t} \in [\mathbf{0}, \mathbf{b}]: \\ |\mathbf{t}|^{1/\alpha} \leq \|x\|}} |\mathbf{t}| \, \mathrm{d}\mathbf{t} \geq \mathrm{const} \cdot \|x\|^{\alpha(N+1)}, \tag{4.30}$$

and the result follows. On the other hand, if $\|x\| > R_0$, then the preceding display still holds uniformly for all $x \in [-M, M]^d$. This proves the proposition. $\qquad\square$

We mention also the following; it is an immediate consequence of Proposition 4.6 and the scaling relation (4.25).

Lemma 4.7. *Choose and fix some $\boldsymbol{b} \in (0, \infty)^N$ and define $[\boldsymbol{0}, \boldsymbol{b}]$ as in Lemma 4.5. Then there exists a constant $A_2 \in (1, \infty)$—depending only on d, N, α, $\min_{1 \leq j \leq N} b_j$, and $\max_{1 \leq j \leq N} b_j$—such that for all $x \in \mathbf{R}^d$,*

$$\int_{[\boldsymbol{0}, 2\boldsymbol{b}]} g_{\boldsymbol{t}}(x) \, \mathrm{d}\boldsymbol{t} \leq A_2 \int_{[\boldsymbol{0}, \boldsymbol{b}]} g_{\boldsymbol{t}}(x) \, \mathrm{d}\boldsymbol{t}. \tag{4.31}$$

Proof. Let $M > 1$ be a constant. If $x \in [-M, M]^d$, then (4.31) follows from Proposition 4.6. And if $\|x\| \geq M$, then, (4.31) holds because of (4.25) and (4.26), together with the well-known fact that g_1 is continuous and strictly positive everywhere; compare with the first line in (4.29). $\qquad \square$

4.3 First part of the proof

Our goal, in this first half, is to prove the following:

$$\mathcal{C}_{d-\alpha N}(E \times F) > 0 \quad \Rightarrow \quad \mathrm{P}\left\{ W(E) \cap \overline{X_\alpha(\mathbf{R}_+^N)} \cap F \neq \varnothing \right\} > 0. \tag{4.32}$$

Because $E \subset (0, \infty)$ and $F \subset \mathbf{R}^d$ are assumed to be compact, there exists a $q \in (1, \infty)$ such that

$$E \subseteq \left[q^{-1}, q\right] \quad \text{and} \quad F \subseteq [-q, q]^d. \tag{4.33}$$

We will use q for this purpose unwaiveringly.

Define

$$f_\epsilon(x) := \frac{1}{\nu_d \epsilon^d} \mathbf{1}_{B_0(\epsilon)}(x) \quad \text{and} \quad \phi_\epsilon(x) := (f_\epsilon * f_\epsilon)(x). \tag{4.34}$$

For every $\mu \in \mathcal{P}(E \times F)$ and $\epsilon > 0$ we define a variable $Z_\epsilon(\mu)$ by,

$$Z_\epsilon(\mu) := \int_{[1,2]^N} \mathrm{d}\boldsymbol{u} \int_{E \times F} \mu(\mathrm{d}s \, \mathrm{d}x) \, \phi_\epsilon(W(s) - x) \phi_\epsilon(X_\alpha(\boldsymbol{u}) - x). \tag{4.35}$$

Lemma 4.8. *There exists a constant $a \in (0, \infty)$ such that*

$$\inf_{\mu \in \mathcal{P}(E \times F)} \inf_{\epsilon \in (0,1)} \mathrm{E}\left[Z_\epsilon(\mu)\right] \geq a. \tag{4.36}$$

Proof. Thanks to the triangle inequality, whenever $u \in B_0(\epsilon/2)$ and $v \in B_0(\epsilon/2)$, then we have also $u - v \in B_0(\epsilon)$ and $v \in B_0(\epsilon)$. Therefore, for all $u \in \mathbf{R}^d$ and $\epsilon > 0$,

$$\begin{aligned}
\phi_\epsilon(u) &= \frac{1}{\nu_d^2 \epsilon^{2d}} \int_{\mathbf{R}^d} \mathbf{1}_{B_0(\epsilon)}(u - v) \mathbf{1}_{B_0(\epsilon)}(v) \, \mathrm{d}v \\
&\geq \frac{1}{\nu_d^2 \epsilon^{2d}} \mathbf{1}_{B_0(\epsilon/2)}(u) \int_{\mathbf{R}^d} \mathbf{1}_{B_0(\epsilon/2)}(v) \, \mathrm{d}v \geq 2^{-d} f_{\epsilon/2}(u).
\end{aligned} \tag{4.37}$$

13

Because $f_{\epsilon/2}$ is a probability density, and since $\epsilon \in (0,1)$, the preceding implies that for all $\boldsymbol{u} \in [1,2]^N$ and $x \in \mathbf{R}^d$,

$$
\begin{aligned}
(\phi_\epsilon * g_{\boldsymbol{u}})(x) &= \int_{\mathbf{R}^d} \phi_\epsilon(u) g_{\boldsymbol{u}}(x-u)\, \mathrm{d}u \\
&\geq 2^{-d} \int_{\mathbf{R}^d} f_{\epsilon/2}(u) g_{\boldsymbol{u}}(x-u)\, \mathrm{d}u \geq 2^{-d} \inf_{\|z-x\| \leq 1/2} g_{\boldsymbol{u}}(z).
\end{aligned}
\tag{4.38}
$$

Since $F \subset [-q,q]^d$, Lemma 4.5 tells us that

$$
a_0 := \inf_{\boldsymbol{u} \in [1,2]^N} \inf_{x \in F} \inf_{\epsilon \in (0,1)} (\phi_\epsilon * g_{\boldsymbol{u}})(x) > 0.
\tag{4.39}
$$

And therefore, for all $\epsilon > 0$ and $\mu \in \mathcal{P}(E \times F)$,

$$
\begin{aligned}
\mathrm{E}\left[Z_\epsilon(\mu)\right] &= \int_{E \times F} \mu(\mathrm{d}s\,\mathrm{d}x)\, (\phi_\epsilon * p_s)(x) \int_{[1,2]^N} \mathrm{d}\boldsymbol{u}\, (\phi_\epsilon * g_{\boldsymbol{u}})(x) \\
&\geq a_0 \int_{E \times F} (\phi_\epsilon * p_s)(x)\, \mu(\mathrm{d}s\,\mathrm{d}x) \geq a_0 \inf_{s \in [1/q,q]} \inf_{x \in F} \inf_{\epsilon \in (0,1)} (\phi_\epsilon * p_s)(x),
\end{aligned}
\tag{4.40}
$$

which is clearly positive. $\qquad\square$

Proposition 4.9. *There exists a constant $b \in (0,\infty)$ such that the following inequality holds simultaneously for all $\mu \in \mathcal{P}(E \times F)$:*

$$
\sup_{\epsilon > 0} \mathrm{E}\left(|Z_\epsilon(\mu)|^2\right) \leq b\mathcal{E}_{d-\alpha N}(\mu).
\tag{4.41}
$$

Proof. First of all, let us note the following compliment to (4.37):

$$
\phi_\epsilon(z) \leq 2^d f_{2\epsilon}(z) \qquad \text{for all } \epsilon > 0 \text{ and } z \in \mathbf{R}^d.
\tag{4.42}
$$

Define, for the sake of notational simplicity,

$$
\mathcal{Q}_\epsilon(t,x;s,y) := \phi_\epsilon(W(t)-x)\phi_\epsilon(W(s)-y).
\tag{4.43}
$$

Next we apply the Markov property to find that for all (t,x) and (s,y) in $E \times F$ such that $s < t$, and all $\epsilon > 0$,

$$
\mathrm{E}\left[\mathcal{Q}_\epsilon(t,x;s,y)\right] = \mathrm{E}\left[\phi_\epsilon(W(s)-y)\phi_\epsilon(\tilde{W}(t-s)+W(s)-x)\right],
\tag{4.44}
$$

where $\tilde{W}$ is a Brownian motion independent of W. An application of (4.42) yields

$$
\begin{aligned}
\mathrm{E}\left[\mathcal{Q}_\epsilon(t,x;s,y)\right] &\leq 4^d \mathrm{E}\left[f_{2\epsilon}(W(s)-y)f_{2\epsilon}(\tilde{W}(t-s)+W(s)-x)\right] \\
&\leq 8^d \mathrm{E}\left[f_{2\epsilon}(W(s)-y)f_{4\epsilon}(\tilde{W}(t-s)-x+y)\right],
\end{aligned}
\tag{4.45}
$$

14

thanks to the triangle inequality. Consequently, we may apply independence and (4.37) to find that

$$\mathrm{E}\left[\mathcal{Q}_\epsilon(t\,,x\,;s\,,y)\right] \leq 8^d \mathrm{E}\left[f_{2\epsilon}(W(s) - y)\right] \cdot \mathrm{E}\left[f_{4\epsilon}(W(t - s) - x + y)\right]$$

$$\leq 32^d \mathrm{E}\left[\phi_{4\epsilon}(W(s) - y)\right] \cdot \mathrm{E}\left[\phi_{8\epsilon}(W(t - s) - x + y)\right] \quad (4.46)$$

$$= 32^d (\phi_{4\epsilon} * p_s)(y) \cdot (\phi_{8\epsilon} * p_{t-s})(x - y).$$

Since $s \in E$, it follows that $s \geq 1/q$ and hence $\sup_{z \in \mathbf{R}^d} p_s(z) \leq p_{1/q}(0)$. Thus,

$$\mathrm{E}\left[\phi_\epsilon(W(t) - x)\phi_\epsilon(W(s) - y)\right] \leq 32^d p_{1/q}(0) \cdot (\phi_{8\epsilon} * p_{t-s})(x - y). \quad (4.47)$$

By symmetry, the following holds for all $(t\,,x), (s\,,y) \in E \times F$ and $\epsilon > 0$:

$$\mathrm{E}\left[\phi_\epsilon(W(t) - x)\phi_\epsilon(W(s) - y)\right] \leq 32^d p_{1/q}(0) \cdot (\phi_{8\epsilon} * p_{|t-s|})(x - y). \quad (4.48)$$

Similarly, we have the following for all $(\boldsymbol{u}\,,x), (\boldsymbol{v}\,,y) \in [1\,,2]^N \times F$ and $\epsilon > 0$:

$$\mathrm{E}\left[\phi_\epsilon(X_\alpha(\boldsymbol{u}) - x)\phi_\epsilon(X_\alpha(\boldsymbol{v}) - y)\right] \leq 32^d K \cdot (\phi_{8\epsilon} * g_{\boldsymbol{u}-\boldsymbol{v}})(x - y), \quad (4.49)$$

where $K := g_{(1/q,\dots,1/q)}(0) < \infty$ by (4.21), and the definition of $g_t(z)$ has been extended to all $\boldsymbol{t} \in \mathbf{R}^N \setminus \{0\}$ by symmetry, viz.,

$$g_{\boldsymbol{t}}(z) := |\boldsymbol{t}|^{-d/\alpha} g_{\mathbf{1}}\left(\frac{z}{|\boldsymbol{t}|^{1/\alpha}}\right) \qquad \text{for all } z \in \mathbf{R}^d \text{ and } \boldsymbol{t} \in \mathbf{R}^N \setminus \{0\}. \quad (4.50)$$

It follows easily from the preceding that $\mathrm{E}(|Z_\epsilon(\mu)|^2)$ is bounded above by a constant multiple of

$$\iint (\phi_{8\epsilon} * p_{|t-s|})(x - y) \left(\int_{[1,2]^{2N}} (\phi_{8\epsilon} * g_{\boldsymbol{u}-\boldsymbol{v}})(x - y)\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{v}\right) \\ \times \mu(\mathrm{d}t\,\mathrm{d}x)\,\mu(\mathrm{d}s\,\mathrm{d}y), \quad (4.51)$$

uniformly for all $\epsilon > 0$. Define

$$\kappa(z) := \int_{[0,1]^N} g_{\boldsymbol{u}}(z)\,\mathrm{d}\boldsymbol{u} \qquad \text{for all } z \in \mathbf{R}^d. \quad (4.52)$$

Then we have shown that, uniformly for every $\epsilon > 0$,

$$\mathrm{E}\left(|Z_\epsilon(\mu)|^2\right) \quad (4.53)$$

$$\leq \text{const} \cdot \iint (\phi_{8\epsilon} * p_{|t-s|})(x - y)(\phi_{8\epsilon} * \kappa)(x - y)\,\mu(\mathrm{d}t\,\mathrm{d}x)\,\mu(\mathrm{d}s\,\mathrm{d}y).$$

It follows easily from (4.21) that the conditions of Proposition 4.2 are met for $\sigma(\mathrm{d}x) := \nu(\mathrm{d}x) := \phi_{8\epsilon}(x)\,\mathrm{d}x$, and therefore that proposition yields the following bound: Uniformly for all $\epsilon > 0$,

$$\mathrm{E}\left(|Z_\epsilon(\mu)|^2\right) \leq \text{const} \cdot \iint p_{|t-s|}(x - y)\kappa(x - y)\,\mu(\mathrm{d}t\,\mathrm{d}x)\,\mu(\mathrm{d}s\,\mathrm{d}y). \quad (4.54)$$

According to Proposition 4.6, $\kappa(z) \leq \text{const}/\|z\|^{d-\alpha N}$ uniformly for all $z \in \{x - y : x, y \in F\}$, and the proof is thus completed. $\qquad\square$

15

Now we establish (4.32).

Proof of Theorem 3.1: First half. If $\mathcal{C}_{d-\alpha N}(E \times F) > 0$, then there exists $\mu_0 \in \mathcal{P}(E \times F)$ such that $\mathcal{E}_{d-\alpha N}(\mu_0) < \infty$, by default. We apply the Paley–Zygmund inequality [10, p. 72] to Lemma 4.8 and Proposition 4.9, with μ replaced by μ_0, to find that for all $\epsilon > 0$ and $\mu \in \mathcal{P}(E \times F)$,

$$\mathrm{P}\left\{Z_\epsilon(\mu_0) > 0\right\} \geq \frac{|\mathrm{E}Z_\epsilon(\mu_0)|^2}{\mathrm{E}\left(|Z_\epsilon(\mu_0)|^2\right)} \geq \frac{a^2/b}{\mathcal{E}_{d-\alpha N}(\mu_0)}. \tag{4.55}$$

If $Z_\epsilon(\mu_0)(\omega) > 0$ for some ω in the underlying sample space, then it follows from (4.35) and (4.37) that

$$\inf_{s \in E} \inf_{x \in F} \inf_{\boldsymbol{u} \in [1,2]^N} \max\left(\|W(s) - x\|\,,\,\|X_\alpha(\boldsymbol{u}) - x\|\right)(\omega) \leq \epsilon, \tag{4.56}$$

for the very same ω. As the right-most term in (4.55) is independent of $\epsilon > 0$, the preceding establishes (4.32); i.e., the first half of the proof of Theorem 3.1. $\square$

4.4 Second part of the proof

For the second half of our proof we aim to prove that for any two positive real numbers $a < b$,

$$\mathrm{P}\left\{W(E) \cap \overline{X_\alpha([a\,,b]^N)} \cap F \neq \varnothing\right\} > 0 \;\Rightarrow\; \mathcal{C}_{d-\alpha N}(E \times F) > 0. \tag{4.57}$$

This would complete our derivation of Theorem 3.1. In order to somewhat simplify the exposition, we will prove the following slightly-weaker statement.

$$\mathrm{P}\left\{W(E) \cap X_\alpha\left(\left[1\,,\tfrac{3}{2}\right]^N\right) \cap F \neq \varnothing\right\} > 0 \;\Rightarrow\; \mathcal{C}_{d-\alpha N}(E \times F) > 0. \tag{4.58}$$

This is so, because $X^{(1)}, \ldots, X^{(N)}$ are right-continuous random functions that possess left-limits. However, we omit the details of this more-or-less routine argument.

Henceforth, we assume that the displayed probability in (4.58) is positive. Let us fix a small positive η, and define E^η and F^η to be the respective closed η-enlargements of E and F. Let ∂ be a point that is not in $\mathbf{R}_+ \times \mathbf{R}_+^N$, and define $\mathbf{Q}_\partial := \mathbf{Q}_+^{N+1} \cup \{\partial\}$. Then, we can always find a random variable $T^\eta := (T_1^\eta, \boldsymbol{t}_2^\eta)$, with values in $\mathbf{Q}_\partial \cap (E^\eta \times [1\,,\tfrac{3}{2}]^N)$ such that:

1. If there exists $(s\,,\boldsymbol{u}) \in E^\eta \times [1\,,\tfrac{3}{2}]^N$ such that $W(s) = X_\alpha(\boldsymbol{u}) \in F^\eta$, then $T^\eta \in \mathbf{Q}_\partial \setminus \{\partial\}$, $X_\alpha(\boldsymbol{t}_2^\eta) \in F^\eta$ and $\|W(T_1^\eta) - X_\alpha(\boldsymbol{t}_2^\eta)\| < \epsilon$, where $\epsilon > 0$ will be chosen as in (4.60) below; and

2. If there is no $(s\,,\boldsymbol{u}) \in E^\eta \times [1\,,\tfrac{3}{2}]^N$ such that $W(s) = X_\alpha(\boldsymbol{u}) \in F^\eta$, then $T^\eta = \partial$.

To prove this, first note that for all fixed $s > 0$ and $\boldsymbol{u} \in (0, \infty)^N$, the joint law of $W(s)$ and $X_\alpha(\boldsymbol{u})$ is absolutely continuous. Therefore, path-regularity tells us that if there exists $(s, \boldsymbol{u}) \in E^\eta \times [1, \frac{3}{2}]^N$ such that $W(s) = X_\alpha(\boldsymbol{u}) \in F^\eta$, then there exists $(t, \boldsymbol{v}) \in E^\eta \times [1, \frac{3}{2}]^N \cap \mathbf{Q}_+^{N+1}$ such that $W(t), X_\alpha(\boldsymbol{v}) \in F^\eta$ and $\|W(t) - X_\alpha(\boldsymbol{v})\| < \epsilon$. The rest follows by an enumeration of the $(N+1)$-dimensional rationals. In order to ensure this is all correct without having to worry about various almost-sure versus sure statements, we are tacitly throwing away a few null sets outside which $X^{(j)}$'s are all right-continuous, and W is continuous. This can and will be done without any loss in generality.

Now for any Borel sets $G_1 \subseteq E^\eta$ and $G_2 \subseteq F^\eta$, define

$$\mu^\eta(G_1 \times G_2) := \mathrm{P}\left\{T_1^\eta \in G_1, \, X_\alpha(\boldsymbol{t}_2^\eta) \in G_2 \,\middle|\, T^\eta \neq \partial\right\}. \tag{4.59}$$

Evidently, μ^η can be extended as a probability measure on $E^\eta \times F^\eta$.

For every $\epsilon > 0$, we define $Z_\epsilon(\mu^\eta)$ by (4.35), but insist on one [important] change. Namely, now, we use the gaussian mollifier,

$$\phi_\epsilon(z) := \frac{1}{(2\pi\epsilon^2)^{d/2}} \exp\left(-\frac{\|z\|^2}{2\epsilon^2}\right), \tag{4.60}$$

in place of $f_\epsilon * f_\epsilon$. [The change in the notation is used only in this portion of the present proof.]

Thanks to the proof of Lemma 4.8,

$$\inf_{\epsilon, \eta \in (0,1)} \mathrm{E}\left[Z_\epsilon(\mu^\eta)\right] > 0. \tag{4.61}$$

Similarly, (4.53) tells us that

$$\sup_{\epsilon, \eta \in (0,1)} \mathrm{E}\left(|Z_\epsilon(\mu^\eta)|^2\right) \tag{4.62}$$

$$\leq \mathrm{const} \cdot \iint (\phi_{8\epsilon} * p_{|t-s|})(x-y)(\phi_{8\epsilon} * \kappa)(x-y)\, \mu^\eta(\mathrm{d}s\,\mathrm{d}x)\, \mu^\eta(\mathrm{d}t\,\mathrm{d}y),$$

where κ is defined by (4.52). Define

$$\tilde{\kappa}(z) := \int_{[0,1/2]^N} g_{\boldsymbol{t}}(z)\, \mathrm{d}\boldsymbol{t} \qquad \text{for all } z \in \mathbf{R}^d. \tag{4.63}$$

Thanks to Lemma 4.7,

$$\sup_{\epsilon, \eta \in (0,1)} \mathrm{E}\left(|Z_\epsilon(\mu^\eta)|^2\right) \tag{4.64}$$

$$\leq \mathrm{const} \cdot \iint (\phi_{8\epsilon} * p_{|t-s|})(x-y)(\phi_{8\epsilon} * \tilde{\kappa})(x-y)\, \mu^\eta(\mathrm{d}s\,\mathrm{d}x)\, \mu^\eta(\mathrm{d}t\,\mathrm{d}y).$$

Now we are ready to explain why we had to change the definition of ϕ_ϵ from $f_\epsilon * f_\epsilon$ to the present gaussian ones: In the present gaussian case, both subscripts

17

of "8ϵ" can be replaced by "ϵ" at no extra cost; see (4.65) below. Here is the reason why:

First of all, note that ϕ_ϵ is still positive definite; in fact, $\hat{\phi}_\epsilon(\xi) = \mathrm{e}^{-\epsilon^2\|\xi\|^2/2} \geq 0$ for all $\xi \in \mathbf{R}^d$. Next—and this is important—we can observe that $\hat{\phi}_\epsilon \leq \hat{\phi}_\delta$ whenever $0 < \delta < \epsilon$. And hence, the following holds, thanks to Remark 4.3:

$$\sup_{\epsilon,\eta\in(0,1)} \mathrm{E}\left(\left|Z_\epsilon(\mu^\eta)\right|^2\right) \tag{4.65}$$

$$\leq \mathrm{const}\cdot\iint (\phi_\epsilon * p_{|t-s|})(x-y)(\phi_\epsilon * \tilde{\kappa})(x-y)\,\mu^\eta(\mathrm{d}s\,\mathrm{d}x)\,\mu^\eta(\mathrm{d}t\,\mathrm{d}y).$$

This proves the assertion that "8ϵ can be replaced by ϵ."

Now define a partial order $\prec$ on $\mathbf{R}^N$ as follows: $\boldsymbol{u} \prec \boldsymbol{v}$ if and only if $u_i \leq v_i$ for all $i = 1, \ldots, N$. Let $\mathcal{X}_{\boldsymbol{v}}$ denote the σ-algebra generated by the collection $\{X_\alpha(\boldsymbol{u})\}_{\boldsymbol{u}\prec\boldsymbol{v}}$. Also define $\mathcal{G} := \{\mathcal{G}_t\}_{t\geq0}$ to be the usual augmented filtration of the Brownian motion W.

According to Theorem 2.3.1 of [10, p. 405], $\{\mathcal{X}_{\boldsymbol{v}}\}$ is a commuting N-parameter filtration [10, p. 233]. Hence, so is the $(N+1)$-parameter filtration

$$\mathcal{F} := \left\{\mathcal{F}_{s,\boldsymbol{u}};\, s \geq 0,\, \boldsymbol{u} \in \mathbf{R}_+^N\right\}, \tag{4.66}$$

where $\mathcal{F}_{s,\boldsymbol{u}} := \mathcal{G}_s \times \mathcal{X}_{\boldsymbol{u}}$.

Now, for any $(s,\boldsymbol{u}) \in \mathbf{Q}^{N+1} \cap (E^\eta \times [1,\frac{3}{2}]^N)$,

$$\mathrm{E}\left[Z_\epsilon(\mu^\eta)\,|\,\mathcal{F}_{s,\boldsymbol{u}}\right] \geq \int_{V(\boldsymbol{u})} \mathrm{d}\boldsymbol{v} \int_{\substack{E^\eta\times F^\eta\\t\geq s}} \mu^\eta(\mathrm{d}t\,\mathrm{d}x)\,\mathcal{T}_\epsilon(t,x\,;\boldsymbol{v}), \tag{4.67}$$

where

$$V(\boldsymbol{u}) := \left\{\boldsymbol{v} \in [1,2]^N:\, u_j \leq v_j\ \text{ for all } 1 \leq j \leq N\right\}, \tag{4.68}$$

and

$$\mathcal{T}_\epsilon(t,x\,;\boldsymbol{v}) := \mathrm{E}\left[\phi_\epsilon(W(t)-x)\phi_\epsilon(X_\alpha(\boldsymbol{v})-x)\,\big|\,\mathcal{F}_{s,\boldsymbol{u}}\right]. \tag{4.69}$$

Thanks to independence, and the respective Markov properties of the processes $W, X^{(1)}, \ldots, X^{(N)}$,

$$\mathcal{T}_\epsilon(t,x\,;\boldsymbol{v}) = \mathrm{E}\left[\phi_\epsilon(W(t)-x)\,\big|\,\mathcal{G}_s\right]\cdot\mathrm{E}\left[\phi_\epsilon(X_\alpha(\boldsymbol{v})-x)\,\big|\,\mathcal{X}_{\boldsymbol{u}}\right]$$
$$= (\phi_\epsilon * p_{t-s})(x - W(s))\cdot(\phi_\epsilon * g_{\boldsymbol{v}-\boldsymbol{u}})(x - X_\alpha(\boldsymbol{u})). \tag{4.70}$$

Therefore, the definition (4.63) of $\tilde{\kappa}$ and the triangle inequality together reveal that with probability one,

$$\mathrm{E}\left[Z_\epsilon(\mu^\eta)\,|\,\mathcal{F}_{s,\boldsymbol{u}}\right]$$
$$\geq \int_{\substack{E^\eta\times F^\eta\\t\geq s}} (\phi_\epsilon * p_{t-s})(x - W(s))(\phi_\epsilon * \tilde{\kappa})(x - X_\alpha(\boldsymbol{u}))\,\mu^\eta(\mathrm{d}t\,\mathrm{d}x), \tag{4.71}$$

for every $\boldsymbol{u} \in [1,3/2]^N$. This inequality is valid almost surely, simultaneously for all rational $s \in E^\eta$ and all $\boldsymbol{u} \in [1,3/2]^N \cap \mathbf{Q}_+^N$. Thus, we can work on the

18

event $\{T^\eta \neq \partial\}$, and replace s by T_1^η and $\boldsymbol{u}$ by $\boldsymbol{t}_2^\eta$ to deduce that almost surely,

$$\mathrm{E}\left[Z_\epsilon(\mu^\eta) \,\big|\, \mathcal{F}_{s,\boldsymbol{u}}\right] \geq \mathbf{1}_{\{T^\eta \neq \partial\}} \cdot \int_{\substack{E^\eta \times F^\eta \\ t \geq s}} \Psi_\epsilon(t,x)\, \mu^\eta(\mathrm{d}t\, \mathrm{d}x), \qquad (4.72)$$

uniformly for all $(s, \boldsymbol{u}) \in \mathbf{Q}_+^{N+1}$, where

$$\Psi_\epsilon(t,x) := (\phi_\epsilon * p_{t-T_1^\eta})(x - W(T_1^\eta))(\phi_\epsilon * \tilde{\kappa})(x - X_\alpha(\boldsymbol{t}_2^\eta)). \qquad (4.73)$$

We square both sides of (4.72) and then apply expectations to both sides in order to obtain the following:

$$\mathrm{E}\left\{\left(\sup_{(s,\boldsymbol{u})\in\mathbf{Q}_+^{N+1}} \mathrm{E}\left[Z_\epsilon(\mu^\eta)\,|\,\mathcal{F}_{s,\boldsymbol{u}}\right]\right)^2\right\}$$

$$\geq \mathrm{P}\left\{T^\eta \neq \partial\right\} \cdot \mathrm{E}\left[\left(\int_{\substack{E^\eta \times F^\eta \\ t \geq s}} \Psi_\epsilon(t,x)\, \mu^\eta(\mathrm{d}t\, \mathrm{d}x)\right)^2 \,\Bigg|\, T^\eta \neq \partial\right]. \qquad (4.74)$$

According to (4.59), the latter conditional expectation is equal to the following:

$$\int \left(\int_{\substack{E^\eta \times F^\eta \\ t \geq s}} (\phi_\epsilon * p_{t-s})(x - W(s))(\phi_\epsilon * \tilde{\kappa})(x - y)\, \mu^\eta(\mathrm{d}t\, \mathrm{d}x)\right)^2 \mu^\eta(\mathrm{d}s\, \mathrm{d}y). \qquad (4.75)$$

Since, for $s = T_1^\eta$ and $y = X_\alpha(\boldsymbol{t}_2^\eta)$ we have $\|W(s) - y\| < \epsilon$, it can be verified that (4.75) is bounded from below by a constant multiple of

$$\int \left(\int_{\substack{E^\eta \times F^\eta \\ t \geq s}} (\phi_{2\epsilon/3} * p_{t-s})(x - y)(\phi_\epsilon * \tilde{\kappa})(x - y)\, \mu^\eta(\mathrm{d}t\, \mathrm{d}x)\right)^2 \mu^\eta(\mathrm{d}s\, \mathrm{d}y). \quad (4.76)$$

Because of the Cauchy–Schwarz inequality, the quantity in (4.76) is at least

$$\left(\int \int_{\substack{E^\eta \times F^\eta \\ t \geq s}} (\phi_{2\epsilon/3} * p_{t-s})(x - y)(\phi_\epsilon * \tilde{\kappa})(x - y)\, \mu^\eta(\mathrm{d}t\, \mathrm{d}x)\, \mu^\eta(\mathrm{d}s\, \mathrm{d}y)\right)^2, \quad (4.77)$$

which is, in turn, greater than or equal to

$$\frac{1}{4}\left(\int\!\!\int (\phi_{2\epsilon/3} * p_{|t-s|})(x - y)(\phi_\epsilon * \tilde{\kappa})(x - y)\, \mu^\eta(\mathrm{d}t\, \mathrm{d}x)\, \mu^\eta(\mathrm{d}s\, \mathrm{d}y)\right)^2, \qquad (4.78)$$

by symmetry. By using Remark 4.3 again, we see that (4.78) is at least

$$\frac{1}{4}\left(\int\!\!\int (\phi_\epsilon * p_{|t-s|})(x - y)(\phi_\epsilon * \tilde{\kappa})(x - y)\, \mu^\eta(\mathrm{d}t\, \mathrm{d}x)\, \mu^\eta(\mathrm{d}s\, \mathrm{d}y)\right)^2, \qquad (4.79)$$

19

The preceding estimates from below the conditional expectation in (4.74). And this yields a bound on the right-hand side of (4.74). We can also obtain a good estimate for the left-hand side of (4.74). Indeed, the $(N+1)$-parameter filtration $\mathcal{F}$ is commuting; therefore, according to Cairoli's strong $(2,2)$ inequality [10, Theorem 2.3.2, p. 235],

$$\mathrm{E}\left\{\left(\sup_{(s,\boldsymbol{u})\in\mathbf{Q}_+^{N+1}}\mathrm{E}\left[\,Z_\epsilon(\mu^\eta)\,|\,\mathcal{F}_{s,\boldsymbol{u}}\,\right]\right)^2\right\}\le 4^{N+1}\mathrm{E}\left(\left|Z_\epsilon(\mu^\eta)\right|^2\right), \qquad (4.80)$$

and this is in turn at most a constant times the final quantity in (4.79); compare with (4.65). In this way, we are led to the following inequality:

$$\mathrm{P}\left\{T^\eta\ne\partial\right\} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.81)$$
$$\le \mathrm{const}\cdot\left[\iint(\phi_\epsilon*p_{|t-s|})(x-y)(\phi_\epsilon*\tilde{\kappa})(x-y)\,\mu^\eta(\mathrm{d}t\,\mathrm{d}x)\,\mu^\eta(\mathrm{d}s\,\mathrm{d}y)\right]^{-1}.$$

Since the implied constant is independent of ϵ and η, we can let $\epsilon\downarrow 0$. As the integrand is lower semicontinuous, we obtain the following from simple real-variables considerations:

$$\mathrm{P}\left\{T^\eta\ne\partial\right\} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.82)$$
$$\le \mathrm{const}\cdot\left[\iint p_{|t-s|}(x-y)\kappa(x-y)\,\mu^\eta(\mathrm{d}t\,\mathrm{d}x)\,\mu^\eta(\mathrm{d}s\,\mathrm{d}y)\right]^{-1}.$$

Since the term in the reciprocated brackets is identically equal to the energy $\mathcal{E}_{d-\alpha N}(\mu^\eta)$ of μ^η, and because μ^η is a probability measure on $E^\eta\times F^\eta$, we obtain the following:
$$\mathrm{P}\left\{T^\eta\ne\partial\right\}\le\mathrm{const}\cdot\mathcal{C}_{d-\alpha N}(E^\eta\times F^\eta). \qquad (4.83)$$

Once again, we emphasize that the implied constant does not depend on $\eta>0$. As we let $\eta\downarrow 0$, the left-most quantity converges down to the probability that $W(E)\cap F\cap\overline{X_\alpha([1,\frac{3}{2}]^N)}\ne\varnothing$. And the outer regularity of the capacity form $\mathcal{C}_\gamma$ [10, Lemma 2.1.1, p. 534] implies that $\mathcal{C}_{d-\alpha N}(E^\eta\times F^\eta)$ converges down to $\mathcal{C}_{d-\alpha N}(E\times F)$. Therefore, for the same implied constant as in the preceding display,

$$\mathrm{P}\left\{W(E)\cap F\cap\overline{X_\alpha\left([1,\tfrac{3}{2}]^N\right)}\right\}\le\mathrm{const}\cdot\mathcal{C}_{d-\alpha N}(E\times F). \qquad (4.84)$$

This yields (4.58), and hence Theorem 3.1. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

5   Proof of Theorem 1.1

Choose and fix an $\alpha \in (0\,,1)$, and define X_α to be a symmetric stable process in $\mathbf{R}$ with index α. That is, X_α is the same process as X_α specialized to $N = d = 1$. As before, we denote the transition probabilities of X_α by

$$g_t(x) := \frac{\mathrm{P}\{X_\alpha(t) \in \mathrm{d}x\}}{\mathrm{d}x} = \frac{1}{\pi} \int_0^\infty \cos(\xi|x|)\mathrm{e}^{-t\xi^\alpha/2}\,\mathrm{d}\xi. \qquad (5.1)$$

We define υ to be the corresponding *1-potential density*. That is,

$$\upsilon(x) := \int_0^\infty g_t(x)\mathrm{e}^{-t}\,\mathrm{d}t. \qquad (5.2)$$

It is known that for all $m > 0$ there exists $c_m = c_{m,\alpha} > 1$ such that

$$c_m^{-1}|x|^{\alpha-1} \le \upsilon(x) \le c_m|x|^{\alpha-1} \qquad \text{if } |x| \le m; \qquad (5.3)$$

see [10, Lemma 3.4.1, p. 383]. Since $\alpha \in (0\,,1)$, the preceding remains valid even when $x = 0$, as long as we recall that $1/0 := \infty$.

The following forms the first step toward our proof of Theorem 1.1.

Lemma 5.1. *Suppose there exists a $\mu \in \mathcal{P}(E \times F)$ such that $\mathcal{I}_{d+2(1-\alpha)}(\mu)$ is finite, where*

$$\mathcal{I}_\beta(\mu) := \iint \frac{\mathrm{e}^{-\|x-y\|^2/(2|t-s|)}}{|t - s|^{\beta/2}}\,\mu(\mathrm{d}s\,\mathrm{d}x)\,\mu(\mathrm{d}t\,\mathrm{d}y) \qquad \text{for } \beta > 0. \qquad (5.4)$$

Then, the random set $E \cap W^{-1}(F)$ intersects the closure of $X_\alpha(\mathbf{R}_+)$ with positive probability.

Remark 5.2. It is possible, but significantly harder, to prove that the sufficient condition of Lemma 5.1 is also necessary. We will omit the proof of that theorem, since we will not need it. $\qquad\square$

Proof. For all fixed $\epsilon > 0$ and probability measures μ on $(0\,,\infty) \times \mathbf{R}^d$, we define the following parabolic version of (4.35), using the same notation for $\phi_\epsilon := f_\epsilon * f_\epsilon$, etc.:

$$Y_\epsilon(\mu) := \int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}t \int \mu(\mathrm{d}s\,\mathrm{d}x)\,\phi_\epsilon(W(s) - x)\phi_\epsilon(X_\alpha(t) - s). \qquad (5.5)$$

Just as we did in Lemma 4.8, we can find a constant $c \in (0\,,\infty)$—depending only on the geometry of E and F—such that uniformly for all $\mu \in \mathcal{P}(E \times F)$ and $\epsilon \in (0\,,1)$,

$$\mathrm{E}\left[Y_\epsilon(\mu)\right] = \int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}t \int \mu(\mathrm{d}s\,\mathrm{d}x)\,(\phi_\epsilon * p_s)(x)(\phi_\epsilon * g_t)(s) \ge c; \qquad (5.6)$$

but now we apply (5.3) in place of Lemma 4.5.

And we proceed, just as we did in Proposition 4.9, and prove that

$$\mathrm{E}\left(|Y_\epsilon(\mu)|^2\right) \le \mathrm{const} \cdot \mathcal{I}_{d+2(1-\alpha)}(\mu). \qquad (5.7)$$

The only differences between the proof of (5.7) and that of Proposition 4.9 are the following:

- Here we appeal to Proposition 4.4, whereas in Proposition 4.9 we made use of Proposition 4.2; and

- we apply (5.3) in place of both Proposition 4.6 and Lemma 4.7. Otherwise, the details of the two computations are essentially exactly the same.

Lemma 5.1 follows from another application of the Paley–Zygmund lemma [10, p. 72] to (5.6) and (5.7); the Paley–Zygmund lemma is used in a similar way as in the proof of the first half of Theorem 3.1. We omit the details, since this is a standard second-moment computation. $\qquad\square$

Next, we present measure-theoretic conditions that are respectively sufficient and necessary for $\mathcal{I}_{d+2(1-\alpha)}(\mu)$ to be finite for some Borel space-time probability measure μ on $E \times F$.

Lemma 5.3. *We always have*

$$\dim_{\mathrm{H}}(E \times F\,;\varrho) \le 2 \sup\left\{\beta > 0:\ \inf_{\mu \in \mathcal{P}(E \times F)} \mathcal{I}_\beta(\mu) < \infty\right\}. \qquad (5.8)$$

Proof. For all space-time probability measures μ, and $\tau > 0$ define the *space-time τ-dimensional Bessel–Riesz energy* of μ as

$$\Upsilon_\tau(\mu\,;\varrho) := \iint \frac{\mu(\mathrm{d}s\,\mathrm{d}x)\,\mu(\mathrm{d}t\,\mathrm{d}y)}{[\varrho((s\,,x)\,;(t\,,x))]^\tau}. \qquad (5.9)$$

A suitable formulation of Frostman's theorem [18] implies that

$$\dim_{\mathrm{H}}(E \times F\,;\varrho) = \sup\left\{\tau > 0:\ \Upsilon_\tau(\mu\,;\varrho) < \infty\right\}. \qquad (5.10)$$

We can consider separately the cases that $\|x - t\|^2 \le |s - t|$ and $\|x - y\|^2 > |s - t|$, and hence deduce that

$$\frac{\mathrm{e}^{-\|x-y\|^2/(2|t-s|)}}{|s - t|^\beta} \le \min\left(\frac{c}{\|x - y\|^{2\beta}}\,,\ \frac{1}{|s - t|^\beta}\right), \qquad (5.11)$$

where $c := \sup_{z>1} z^{2\beta}\mathrm{e}^{-z/2}$ is finite. Consequently, $\mathcal{I}_\beta(\mu) \le c'\Upsilon_{2\beta}(\mu\,;\varrho)$, with $c' := \max(c\,,1)$, and (5.8) follows from (5.10). $\qquad\square$

Lemma 5.4. *With probability one,*

$$\dim_{\mathrm{H}}\left(E \cap W^{-1}(F)\right) \le \frac{\dim_{\mathrm{H}}(E \times F\,;\varrho) - d}{2}. \qquad (5.12)$$

22

Proof. Choose and fix some $r > 0$. Let $\mathcal{T}(r)$ denote the collection of all intervals of the form $[t - r^2, t + r^2]$ that are in $[1/q, q]$. Also, let $\mathcal{S}(r)$ denote the collection of all closed Euclidean $[\ell^2]$ balls of radius r that are contained in $[-q, q]^d$. Recall that X_α is a symmetric stable process of index $\alpha \in (0, 1)$ that is independent of W. It is well known that uniformly for all $r \in (0, 1)$,

$$\sup_{I \in \mathcal{T}(r)} \mathrm{P}\left\{ X_\alpha([0, 1]) \cap I \neq \varnothing \right\} \leq \text{const} \cdot r^{2(1-\alpha)}; \tag{5.13}$$

see [10, Lemma 1.4.3., p. 355], for example. It is just as simple to prove that the following holds uniformly for all $r \in (0, 1)$:

$$\sup_{I \in \mathcal{T}(r)} \sup_{J \in \mathcal{S}(r)} \mathrm{P}\left\{ W(I) \cap J \neq \varnothing \right\} \leq \text{const} \cdot r^d. \tag{5.14}$$

[Indeed, conditional on $\{ W(I) \cap J \neq \varnothing \}$, the random variable $W(t)$ comes to within r of J with a minimum positive probability, where t denotes the smallest point in I.] Because $W(I) \cap J \neq \varnothing$ if and only if $W^{-1}(J) \cap I \neq \varnothing$, it follows that uniformly for all $r \in (0, 1)$,

$$\sup_{I \in \mathcal{T}(r)} \sup_{J \in \mathcal{S}(r)} \mathrm{P}\left\{ W^{-1}(J) \cap I \cap X_\alpha([0, 1]) \neq \varnothing \right\} \leq \text{const} \cdot r^{d + 2(1-\alpha)}. \tag{5.15}$$

Define

$$\mathcal{R} := \bigcup_{r \in (0, 1)} \{ I \times J : \ I \in \mathcal{T}(r) \text{ and } J \in \mathcal{S}(r) \}. \tag{5.16}$$

Thus, $\mathcal{R}$ denotes the collection of all "space-time parabolic rectangles" whose ϱ-diameter lies in the interval $(0, 1)$.

Suppose $d + 2(1 - \alpha) > \dim_{\mathrm{H}}(E \times F; \varrho)$. By the definition of Hausdorff dimension, and a Vitali-type covering argument—see Mattila [13, Theorem 2.8, p. 34]—for all $\epsilon > 0$ we can find a countable collection $\{ E_j \times F_j \}_{j=1}^\infty$ of elements of $\mathcal{R}$ such that: (i) $\cup_{j=1}^\infty (E_j \times F_j)$ contains $E \times F$; (ii) The ϱ-diameter of $E_j \times F_j$ is positive and less than one [strictly] for all $j \geq 1$; and (iii) $\sum_{j=1}^\infty |\varrho\text{-diam}(E_j \times F_j)|^{d+2(1-\alpha)} \leq \epsilon$. Thanks to (5.15),

$$\mathrm{P}\left\{ W^{-1}(F) \cap E \cap X_\alpha([0, 1]) \neq \varnothing \right\} \leq \sum_{j=1}^\infty \mathrm{P}\left\{ W^{-1}(F_j) \cap E_j \cap X_\alpha([0, 1]) \neq \varnothing \right\}$$

$$\leq \text{const} \cdot \sum_{j=1}^\infty |\varrho\text{-diam}(E_j \times F_j)|^{d+2(1-\alpha)} \leq \text{const} \cdot \epsilon. \tag{5.17}$$

Since neither the implied constant nor the left-most term depend on the value of ϵ, the preceding shows that $W^{-1}(F) \cap E \cap X_\alpha([0, 1])$ is empty almost surely.

Now let us recall half of McKean's theorem [10, Example 2, p. 436]: *If* $\dim_{\mathrm{H}}(A) > 1 - \alpha$, *then* $X_\alpha([0, 1]) \cap A$ *is nonvoid with positive probability.* We apply McKean's theorem, conditionally, with $A := W^{-1}(F) \cap E$ to find that if $d + 2(1 - \alpha) > \dim_{\mathrm{H}}(E \times F; \varrho)$, then

$$\dim_{\mathrm{H}}\left(W^{-1}(F) \cap E \right) \leq 1 - \alpha \qquad \text{almost surely.} \tag{5.18}$$

The preceding is valid almost surely, simultaneously for all rational values of $1 - \alpha$ that are strictly between one and $\frac{1}{2}(\dim_{\mathrm{H}}(E \times F \,; \varrho) - d)$. Thus, the result follows. $\qquad\square$

Proof of Theorem 1.1. By the modulus of continuity of Brownian motion, there exists a null set off which $\dim_{\mathrm{H}} W(A) \le 2 \dim_{\mathrm{H}} A$, simultaneously for all Borel sets $A \subseteq \mathbf{R}_+$ that might—or might not—depend on the Brownian path itself. Since $W(E \cap W^{-1}(F)) = W(E) \cap F$, Lemma 5.4 implies that

$$\dim_{\mathrm{H}}(W(E) \cap F) \le \dim_{\mathrm{H}}(E \times F \,; \varrho) - d \qquad \text{almost surely.} \qquad (5.19)$$

For the remainder of the proof we assume that $d \ge 2$, and propose to prove that

$$\|\dim_{\mathrm{H}}(W(E) \cap F)\|_{L^\infty(\mathrm{P})} \ge \dim_{\mathrm{H}}(E \times F \,; \varrho) - d. \qquad (5.20)$$

Henceforth, we assume without loss of generality that

$$\dim_{\mathrm{H}}(E \times F \,; \varrho) > d; \qquad (5.21)$$

for there is nothing left to prove otherwise. In accord with the theory of Taylor and Watson [18], (5.21) implies that $\mathrm{P}\{W(E) \cap F \ne \varnothing\} > 0$.

According to Kaufman's uniform-dimension theorem [7], the Hausdorff dimension of $W(E) \cap F$ is almost surely equal to twice the Hausdorff dimension of $E \cap W^{-1}(F)$. Therefore, it suffices to prove the following in the case that $d \ge 2$:

$$\left\|\dim_{\mathrm{H}}\left(E \cap W^{-1}(F)\right)\right\|_{L^\infty(\mathrm{P})} \ge \frac{\dim_{\mathrm{H}}(E \times F \,; \varrho) - d}{2}, \qquad (5.22)$$

as long as the right-hand side is positive. If $\alpha \in (0\,,1)$ satisfies

$$1 - \alpha < \frac{\dim_{\mathrm{H}}(E \times F \,; \varrho) - d}{2}, \qquad (5.23)$$

than Lemma 5.3 implies that $\mathcal{I}_{d+2(1-\alpha)}(\mu) < \infty$ for some $\mu \in \mathcal{P}(E \times F)$. Thanks to Lemma 5.1, $E \cap W^{-1}(F) \cap \overline{X_\alpha([0\,,1])} \ne \varnothing$ with positive probability. Consequently,

$$\mathrm{P}\left\{\dim_{\mathrm{H}}(E \cap W^{-1}(F)) \ge 1 - \alpha\right\} > 0, \qquad (5.24)$$

because the second half of McKean's theorem implies that *if $\dim_{\mathrm{H}}(A) < 1 - \alpha$, then $\overline{X_\alpha(\mathbf{R}_+)} \cap A = \varnothing$ almost surely.* Since (5.24) holds for all $\alpha \in (0\,,1)$ that satisfy (5.23), (5.22) follows. This completes the proof. $\qquad\square$

Remark 5.5. Let us mention the following byproduct of our proof of Theorem 1.1: For every $d \ge 1$,

$$\left\|\dim_{\mathrm{H}}\left(E \cap W^{-1}(F)\right)\right\|_{L^\infty(\mathrm{P})} = \frac{\dim_{\mathrm{H}}(E \times F \,; \varrho) - d}{2}. \qquad (5.25)$$

When $d = 1$, this was found first by Kaufman [8], who used other arguments [for the harder half]. See Hawkes [6] for similar results in case W is replaced by a stable subordinator of index $\alpha \in (0\,,1)$. $\qquad\square$

We conclude this paper with some problems that continue to elude us.

Open Problems. Theorem 1.2 and 1.1 together imply that when $d \geq 2$,

$$\sup\left\{\gamma > 0 : \inf_{\mu \in \mathcal{P}(E \times F)} \mathcal{E}_\gamma(\mu) < \infty\right\} = \dim_{\mathrm{H}}(E \times F \,;\rho) - d. \qquad (5.26)$$

The preceding is a kind of "parabolic Frostman theorem." And we saw in the introduction that (5.26) is in general false when $d = 1$. We would like to better understand why the one-dimensional case is so different from the case $d \geq 2$. Thus, we are led naturally to a number of questions, two of which we state below:

P1. Equation (5.26) is, by itself, a theorem of geometric measure theory. Therefore, we ask, "*Is there a direct proof of* (5.26) *that does not involve random processes, broadly speaking, and Kaufman's uniform-dimension theorem* [7], *in particular*"?

P2. When $d \geq 2$, (5.26) gives an interpretation of the capacity form on the left-hand side of (5.26) in terms of the geometric object on the right-hand side. Can we understand the left-hand side of (5.26) geometrically in the case that $d = 1$?

References

[1] Blumenthal, R. M. and R. K. Getoor, Dual processes and potential theory, In: *Proc. Twelfth Biennial Sem. Canad. Math. Congr. on Time Series and Stochastic Processes; Convexity and Combinatorics* (Vancouver, B.C., 1969) pp. 137–156, Canad. Math. Congr., Montreal, Que., 1970.

[2] Blumenthal, R. M. and R. K. Getoor, *Markov Processes and Potential Theory*, Academic Press, New York-London, 1968.

[3] Doob, Joseph L., *Classical Potential Theory and its Probabilistic Counterpart*, Springer Verlag, Berlin, 2002.

[4] Foondun, Mohammud and Davar Khoshnevisan, Lévy processes, the stochastic heat equation with spatially-colored forcing, and intermittence, *preprint* (2009).

[5] Hawkes, John, Trees generated by a simple branching process, *J. London Math. Soc. (2)* **24** (1981) no. 2, 373–384.

[6] Hawkes, John, Measures of Hausdorff type and stable processes, *Mathematika* **25** (1978) 202–212.

[7] Kaufman, Robert, Une proprieté métrique du mouvement brownien, C. R. Acad. Sci. Paris **268** (1969) 727-728.

[8] Kaufman, Robert, Measures of Hausdorff-type, and Brownian motion, *Mathematika* **19** (1972) 115–119.

[9] Kaufman, Robert and Jan Wei Wu, Parabolic potential theory, *J. Differential Equations* **43**(2) (1982) 204–234.

[10] Khoshnevisan, Davar, *Multiparameter Processes*, Springer Verlag, New York, 2002.

[11] Khoshnevisan, Davar and Yimin Xiao, Harmonic analysis of additive Lévy processes, *Probab. Theory and Related Fields* **145** (2009) 459–515.

[12] Lyons, Russell, Random walks and percolation on trees, *Ann. Probab.* **18** (1990) no. 3, 931–958.

[13] Mattila, Pertti, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, Cambridge, 1995.

[14] Peres, Yuval, *Probability on Trees*, In: Lectures on Probability Theory and Statistics (Saint-Flour, 1997), 193–280, Lecture Notes in Math. **1717**, Springer, Berlin, 1999.

[15] Peres, Yuval, Remarks on intersection-equivalence and capacity-equivalence, *Ann. Inst. H. Poincaré: Phys. Théor.* **64** (1996) no. 3, 339–347.

[16] Orey, Steven, Polar sets for processes with stationary independent increments, In: *Markov Processes and Potential Theory* (Proc. Sympos. Math. Res. Center, Madison, Wis., 1967) pp. 117–126, Wiley, New York, 1967.

[17] Taylor, S. J., Multiple points for the sample paths of the symmetric stable process, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **5** (1966) 247–264.

[18] Taylor, S. J. and N. A. Watson, A Hausdorff measure classification of polar sets for the heat equation, *Math. Proc. Cambridge Philos. Soc.* **97**(2) (1985) 325–344.

[19] Watson, N. A., Thermal capacity, *Proc. London Math. Soc. (3)* **37**(2) (1978) 342–362.

[20] Watson, N. A., Green functions, potentials, and the Dirichlet problem for the heat equation, *Proc. London Math. Soc. (3)* **33**(2) (1976) 251–298 [corrigendum: *Proc. London Math. Soc. (3)* **37**(1) (1978) 32–34.]

Davar Khoshnevisan

Department of Mathematics, University of Utah, Salt Lake City, UT 84112-0090

Email: davar@math.utah.edu

URL: http://www.math.utah.edu/~davar

Yimin Xiao

Department of Statistics and Probability, A-413 Wells Hall, Michigan State University, East Lansing, MI 48824

Email: xiao@stt.msu.edu

URL: http://www.stt.msu.edu/~xiaoyimi