# Tangent Spaces, transversality, Lie bracket

## Tangent spaces and vector bundles

1. Let $X \subset \mathbb{R}^N$ be a submanifold. Show that

$$\{(x, v) \in X \times \mathbb{R}^N \mid v \in T_x(X) \subset \mathbb{R}^N\}$$

is a manifold diffeomorphic to $T(X)$. This is how G-P define the tangent bundle.

2. Recall that $\mathbb{R}P^n$ can be viewed as the space of lines through 0 in $\mathbb{R}^{n+1}$. Let

$$E = \{(x, \ell) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \mid x \in \ell\}$$

Show that the projection $\pi : E \to \mathbb{R}P^n$ to the first coordinate is a line bundle (i.e. it is a 1-dimensional smooth vector bundle). Further show that this bundle is nontrivial.

3. Show that $T(X \times Y)$ is diffeomorphic to $TX \times TY$. Hint: Define a natural map $T(X \times Y) \to TX \times TY$ and then check that it is a diffeomorphism in charts.

4. Show that if a sphere $S^k$ admits a nowhere vanishing vector field then the antipodal map is homotopic to the identity.

5. Show that every smooth map $S^1 \to S^k$ is nullhomotopic if $k > 1$. Note: This is also true for continuous maps. You can prove it using Problem 11 below, although I am not assigning it.

## Lie bracket

6. Let $V = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$ and $W = \frac{\partial}{\partial y}$ be two vector fields on $\mathbb{R}^3$. Compute $[V, W]$ and show that $[V, W](p)$ does not belong to the span of $V(p)$ and $W(p)$ for any $p \in \mathbb{R}^3$.

7. A real vector space $V$ with a bilinear pairing $[\cdot, \cdot] : V \times V \to \mathbb{R}$ is called a *Lie algebra* if the pairing satisfies:

   (i) (alternating) $[v, w] = -[w, v]$ for $v, w \in V$, and

   (ii) (Jacobi identity) $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$ for $u, v, w \in V$.

Show that the vector space of all vector fields on a manifold $X$ is a Lie algebra with respect to the Lie bracket operation. Note: This Lie algebra is infinite dimensional, but we will later see that the vector space of *left invariant* vector fields on a Lie group is a finite dimensional Lie algebra.

## Transversality

8. Let $Z \subset \mathbb{R}^3$ be a submanifold, closed as a subset. Assume $0 \notin Z$. Show that almost every 1-dimensional linear subspace $\ell \subset \mathbb{R}^3$ is transverse to $Z$. Hint: Show that $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, $(t, x, y, z) \to (tx, ty, tz)$ is transverse to $Z$. This is a special case of GP #7 in the transversality section, but they forgot to assume $0 \notin Z$. Can you find a counterexample if $0 \in Z$?

9. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be smooth, $n > 1$, $K \subset \mathbb{R}^n$ compact, $\epsilon > 0$. Show that there is a smooth map $g : \mathbb{R}^n \to \mathbb{R}^n$ which approximates $f$ in the sense that $\|g(x) - f(x)\| < \epsilon$ for all $x \in K$ and so that $dg_x \neq 0$ for any $x \in \mathbb{R}^n$. Hint: Consider $\mathbb{R}^n \times M(n) \to M(n)$, $(x, A) \mapsto df_x + A$ and show it is transverse to $\{0\}$. What goes wrong if $n = 1$?

10. Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ smooth. Show that for almost all $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$ the map $f_a : U \to \mathbb{R}$, $f_a(x_1, \cdots, x_n) = f(x_1, \cdots, x_n) + a_1 x_1 + \cdots + a_n x_n$ is Morse. Note: This is the main input in the proof that smooth functions to $\mathbb{R}$ can be approximated by Morse functions. The general case involves Fubini's theorem. This problem is also from GP, but there is a typo there.

## Tubular neighborhoods

11. Let $Y \subset \mathbb{R}^N$ be a compact submanifold. Show that there is some $\epsilon > 0$ so that if two continuous maps $f, g : X \to Y$ are $\epsilon$-close (i.e. $\|f(x) - g(x)\| < \epsilon$ for every $x \in X$), then $f$ and $g$ are homotopic, i.e. there is a continuous map $H : X \times [0, 1] \to Y$ such that $H_0 = f$, $H_1 = g$. Moreover, if $f$ and $g$ are smooth, then $H$ can be taken to be smooth.

12. Let $X \subset \mathbb{R}^N$ be a hypersurface, i.e. a codimension 1 submanifold. A point $y \in \mathbb{R}^N$ is a *focal point* of $X$ if it is a critical value of the normal bundle map $h : N(X) \to \mathbb{R}^N$, $h(x, v) = x + v$. Compute the set of focal points of the parabola

$$X = \{(x, x^2) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$$

Hint: I am getting $\{(-4t^3, 3t^2 + \frac{1}{2}) \mid t \in \mathbb{R}\}$.