Partitions of unity and transversality

Partitions of unity

1. Let X be a manifold and $A \subset X$ a subset. Suppose that a function $f : A \to \mathbb{R}$ is *locally smoothly extendable*, i.e. for every $a \in A$ there is a neighborhood U_a of a and a smooth function $f_a : U_a \to \mathbb{R}$ such that f and f_a agree on $U_a \cap A$. Show that f *smoothly extendable*, i.e. there is an open set $U \supseteq A$ and a smooth function $\tilde{f} : U \to \mathbb{R}$ that extends f.

2. Let X be a manifold, $A \subset X$ a closed subset and U an open set containing A. Let $f : U \to \mathbb{R}$ be smooth. Show that there is a smooth function $g : X \to \mathbb{R}$ that agrees with f on A. This is called the *Extension Lemma*.

3. Prove the following improvement of the smooth approximation theorem from class. Let X be a manifold, $\epsilon : X \to (0, \infty)$ a continuous function and $f : X \to \mathbb{R}^k$ a continuous function. Then there is a smooth function $g : X \to \mathbb{R}^k$ such that $\|f(x) - g(x)\| < \epsilon(x)$ for every $x \in X$.

4. When $U \subset S^n$ is an open set, by a *vector field* on U we mean a smooth function $v : U \to \mathbb{R}^{n+1}$ such that for every $x \in U$ the vector $v(x) \in \mathbb{R}^{n+1}$ is tangent to $x \in S^n$ (i.e. $v(x)$ and x are perpendicular). Show that for any compact subset $K \subset U$ and any vector field v on U there is a vector field on all of S^n that agrees with v on K.

Transversality

5. Show that the subspaces of symmetric matrices and of skew-symmetric matrices are transverse in the space $M(n)$ of all $n \times n$ matrices.

6. Let X be a manifold. If $A, B \subset X$ are two transverse submanifolds and $x \in A \cap B$ show that $T_x(A \cap B) = T_x(A) \cap T_x(B)$.

7. Show that it is possible for two submanifolds $A, B \subset X$ to have intersection $A \cap B$ a submanifold with $codim(A) + codim(B) = codim(A \cap B)$ even though A, B are *not* transverse to each other. Here $codim(A) = \dim X - \dim A$.

8. For which values of $a > 0$ does the hyperboloid defined by $x^2+y^2-z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = a$ transversally ? What does the intersection look like for different values of a?

9. Let $f : X \to X$ be a smooth self-map of a manifold X. Let

$$Gr(f) = \{(x, f(x)) \in X \times X\}$$

be the graph of f and let $\Delta \subset X \times X$ be the diagonal, i.e. the graph of the identity. Thus $Gr(f) \cap \Delta$ consists of points $(x, x) \in X \times X$ such that $f(x) = x$. Show that $Gr(f) \pitchfork \Delta$ if and only if for every fixed point x of f the derivative $df_x : T_x(X) \to T_x(X)$ does *not* have eigenvalue 1. Hint: The tangent space at $(x, f(x))$ of $Gr(f)$ is the graph of $df_x : T_x(X) \to T_{f(x)}(X)$ as a subspace of $T_{(x,f(x))}(X \times X) = T_x(X) \oplus T_{f(x)}(X)$, which you can prove in charts.

10. Find a counterexample to the Stability Theorem for transversality if compactness is not assumed. That is, find manifolds $X, P, Y \supset Z$ and a smooth map $F : X \times P \to Y$ so that $F_{p_0} \pitchfork Z$ but for every neighborhood U of $p_0 \in P$ there are points $p \in U$ so that F_p is not transverse to Z.